MODULAR SYSTEM

Limits
of Functions

Muhammer Taşkıran

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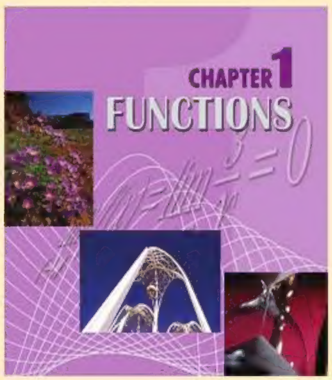
PREFACE

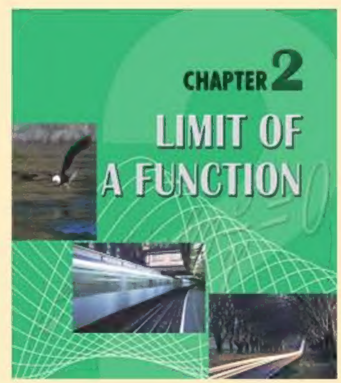
To the Teacher

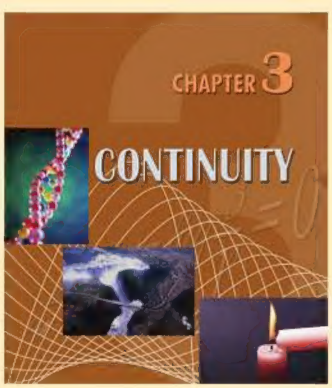
This is an introductory book on mathematical analysis which covers functions, limits and the continuity and discontinuity of functions. All of these topics are necessary prerequisites for the study of integral and differential calculus. The book is divided into three chapters, structured as follows:



Chapter 1 focuses on functions. It begins with an overall review of functions which covers the concepts of domain and range of a function, composite function, inverse function, and even and odd functions. In the second part of the chapter, students study four special types of function: piecewise functions, the absolute value function, the sign function and the floor function.

Chapter 2 builds on the material of the previous chapter and introduces limits. Taking the graphical illustration of a limit as a starting point, the chapter leads the student to an understanding of the concept of limit, including left-hand and right-hand limits and limits of the special functions studied in Chapter 1. Finally, students study the indeterminate forms of limits.



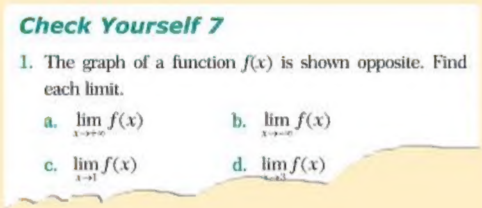


Chapter 3 covers continuity and discontinuity and introduces two important theorems: the Extreme Value Theorem and the Intermediate Value Theorem. Both theorems will be useful when students begin their study of differential calculus.

The book tries to explain the topic as a teacher would explain it in the classroom, giving examples and exercises that prompt students to think for themselves.

Functions and their limits comprise a vast topic, and it is obviously impossible to cover every type of question in the text. However, we believe that we have provided a sufficient range of explanations and examples in the text to enable students to approach most limit problems with confidence.

This book has been designed to be an effective teaching aid, and includes all of the features of the Zambak Modular System high school math teaching series:



Check Yourself 7

1. The graph of a function $f(x)$ is shown opposite. Find each limit.

a. $\lim_{x \to +\infty} f(x)$ b. $\lim_{x \to -\infty} f(x)$

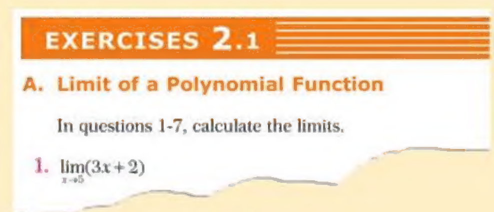
c. $\lim_{x \to 1} f(x)$ d. $\lim f(x)$

Checking Understanding

The book follows a linear teaching approach, with material in the latter sections building on concepts and math covered previously in the text. For this reason, several self-test 'Check Yourself' sections check students' understanding of the material at key points. 'Check Yourself' sections include a rapid answer key that allows students to measure their own performance and understanding. Successful completion of each self-test section allows students to advance to the next topic.

Section Exercises

A number of exercises follow each section. Many of the problems reflect skills or problem-solving techniques encountered in the section. All of these problems can be solved using skills the student should already have mastered.

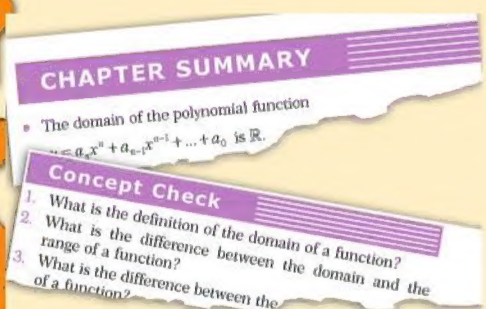


EXERCISES 2.1

A. Limit of a Polynomial Function

In questions 1-7, calculate the limits.

1. $\lim_{x \to 5}(3x+2)$



CHAPTER SUMMARY

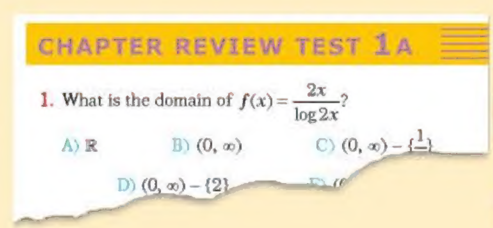
- The domain of the polynomial function $= a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is $\mathbb{R}$.

Concept Check

1. What is the definition of the domain of a function?
2. What is the difference between the domain and the range of a function?
3. What is the difference between the ... of a function?

Review

A chapter summary at the end of each chapter provides a concise review of the main content of the chapter. Included in the summary are a set of concept check questions which ask students to explain key concepts and ideas in their own words.



CHAPTER REVIEW TEST 1A

1. What is the domain of $f(x) = \dfrac{2x}{\log 2x}$?

A) $\mathbb{R}$ B) $(0, \infty)$ C) $(0, \infty) - \{\frac{1}{2}\}$

D) $(0, \infty) - \{2\}$

Testing

Following the chapter summary and concept check, review tests cover material from the chapter and help to prepare students for exams.

Acknowledgements

Many friends and colleagues were of great help in writing this book. I would like to thank everybody who helped me at Zambak Publications, especially Mustafa Kırıkçı. Special thanks also go to Şamil Keskinoğlu for his patient typesetting and design.

Finally, I would like to thank my wife for her support and patience during my work on this book.

Muhammer Taşkıran

CONTENTS

INTRODUCTION

The limit of a function is the most fundamental concept of calculus. Calculus begins with limits. We can define the major ideas of calculus such as continuity, the derivative and integral of a function and the convergence and divergence of sequences and series in terms of limits. This concept of limit makes calculus different from other branches of mathematics such as algebra, geometry, number theory and logic.

The concept of limit evolved from the method of exhaustion, which is a technique used in mathematics to prove theorems regarding the areas and volumes of geometric figures. The method of exhaustion was invented by the Greek mathematician Eudoxus (400 BC). Although it was an ancestor of integral calculus, the method of exhaustion did not use limits and arguments about very tiny quantities. It was instead a completely logical procedure, based on the axiom that a given quantity can be made smaller than another given quantity by successively halving it a finite number of times. This method was also used by Euclid in the fourth century BC and by Archimedes in the third century BC. In their calculation of certain areas, Fermat and Descartes also improved the method. Newton and Leibniz did attempt to explain the concept of limit, but their explanations were not satisfactory. Although Leibniz developed the notation for differential and integral calculus, he never thought of the derivative as a limit. In the ninth century, the mathematician Sabit Bin Kurra made a significant contribution to mathematics with his study of mathematical analysis. His studies are acceptable as a foundation of limits.

Several centuries later, the mathematician d'Alembert made important contributions to our understanding of limits. D'Alembert was one of the first people to understand the importance of functions, and he defined the derivative of a function as the limit of a quotient of growths in 1754. His ideas on limits led him to the test for convergence. In the latter part of his life, d'Alembert turned more towards literature and philosophy. He set out his skepticism concerning metaphysical problems but he accepted the argument in favor of the existence of God based on the belief that intelligence cannot be a product of matter alone.

Concern about the lack of rigorous foundations for calculus grew during the late years of the 18th century. At the beginning of the 18th century, ideas about limits were certainly confusing. The definition of limit that we use today is less than two hundred years old. Before this time, the notions of limit were not clear and only rarely used correctly. In much of his work on calculus, Isaac Newton also failed to acknowledge the fundamental role of limits.

Louis Cauchy was the first mathematician to use what is similar to the epsilon-delta definition of a limit which we use today. In 1821 he gave a calculus course which began with a modern definition of the limit. In his writings, Cauchy used limits as the basis for accurate definitions of continuity, convergence, the derivative and the integral. Cauchy defined the integral of any continuous function on an interval $[a, b]$ to be the limit of the sums of areas of thin rectangles. He attempted to prove that this limit existed for all functions which were continuous on the interval $[a, b]$.

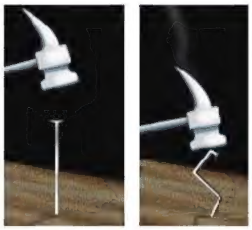
Finally, here is the definition of limit given by Cauchy: if we want all the $f(x)$ values to stay in some small neighborhood around $f(c)$, we simply need to choose a small enough neighborhood for the x-values around c, and prove that we can do this no matter how small the $f(x)$-neighborhood is. In addition, if $f(c)$ is defined then $f(x)$ is continuous at c.

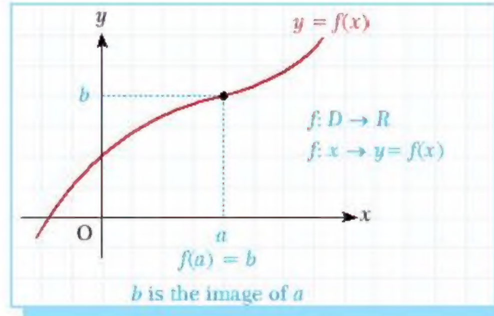
CHAPTER 1
FUNCTIONS

1 TYPES OF FUNCTION

A function from a set D to a set R is a rule that maps each element of D to a single element of R.

$$f\colon \underset{\text{domain}}{\underset{\downarrow}{D}} \to \underset{\text{range}}{\underset{\downarrow}{R}}$$





For each x in D there exists a single element y in R such that $f(x) = y$. x is called the variable of f and $y = f(x)$ is called the image of x. The set of images of all the elements of D is called the image set of f.

Consider the function $f\colon A \to B$, $f(x) = x^2$. We can write this function in different ways: $f(x) = x^2$, $y = x^2$ or $f\colon x \to x^2$. All of these mean the same function. A is the domain and B is the range of f.

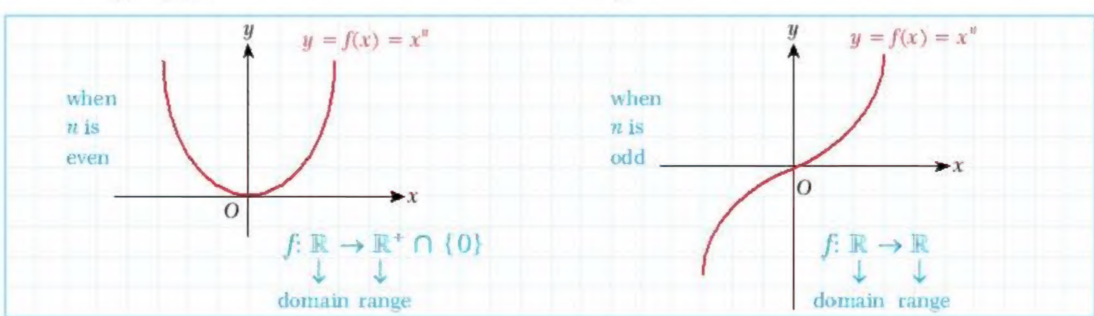
A. DOMAIN AND RANGE OF A FUNCTION

Unless stated otherwise, the domain of a function $f(x)$ is the largest set of real x-values for which $f(x)$ is defined. The range of the function is a set which includes at least all the images of the elements in its domain. The domain and range of many functions are subsets of the set of real numbers. The largest possible range of a function is $\mathbb{R}$.

Let us look at the domain and range of some common types of function. In these examples we will use the letter D to mean the domain of the function. The notation $D(f)$ means the domain of the function f.

Type of function	Form	Domain	Examples
polynomial function	$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ $n \in \mathbb{Z}^+ \cup \{0\}$	$\mathbb{R}$	$f(x) = 2x+5$ $\quad D(f)\colon \mathbb{R}$ $f(x) = 2x^2-3x+1$ $\quad D(f)\colon \mathbb{R}$

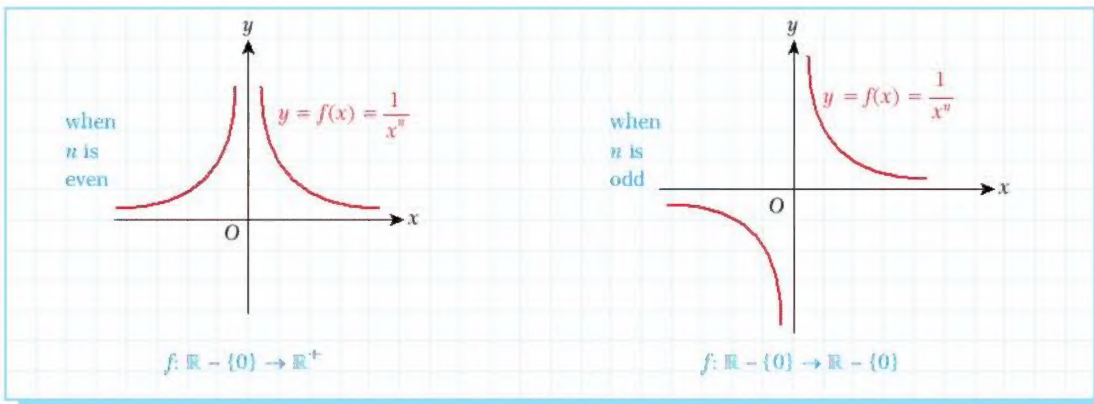
As stated in the table, the domain of any polynomial function f is $\mathbb{R}$. The range of the polynomial function depends on the function itself. For example, let us draw the graph of the function $y = f(x) = x^n$ and find its domain and range when n is odd or even.



Type of function	Form	Domain	Examples
rational function	$f(x) = \dfrac{g(x)}{h(x)}$	$\mathbb{R}-\{x \mid h(x)=0\}$	$f(x) = \dfrac{2x-3}{x+1}$ $D(f)$: $\mathbb{R}-\{-1\}$ $f(x) = \dfrac{x^2+5}{(x-5)(x+2)}$ $D(f)$: $\mathbb{R}-\{5, -2\}$

The value of the denominator in a rational expression cannot be zero, so any numbers which make the denominator zero must be excluded from the domain of a rational function.

As an example, let us look at the graph of the function $y = x^{-n} = \dfrac{1}{x^n}$.

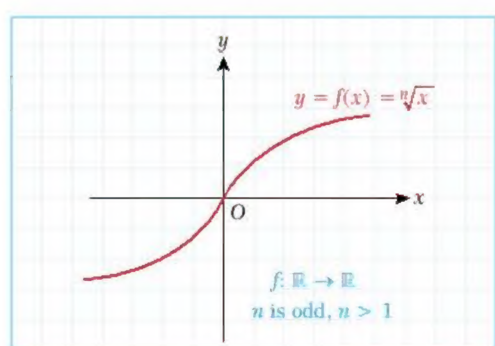


We can see that the domain of a rational function changes according to the function.

Type of function	Form	Domain	Example
radical function	$f(x) = \sqrt[n]{g(x)}$ n is an odd integer	$\mathbb{R}$	$f(x) = \sqrt[3]{x^2+5x}$ $D(f)$: $\mathbb{R}$

When the index of a radical expression is odd, the radicand can be negative, positive or zero. Therefore there is no restriction on the value of x and so the domain is $\mathbb{R}$.

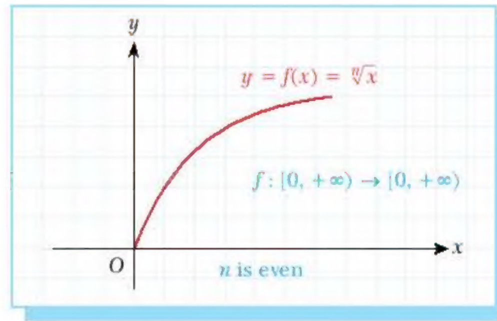
The figure at the right shows the graph of $y = \sqrt[n]{x}$ when n is odd and $n > 1$. We can see that the range of the function is the set of real numbers.



Type of function	Form	Domain	Examples
radical function	$f(x) = \sqrt[n]{g(x)}$ n is an even integer	$\mathbb{R} - \{x \mid g(x) < 0\}$	$f(x) = \sqrt{x^2 - 2}$ $\quad D(f)$: $\mathbb{R} - (-\sqrt{2}, \sqrt{2})$ $f(x) = \sqrt[4]{x^2 - 3x + 2}$ $\quad D(f)$: $\mathbb{R} - (1, 2)$

When the index of a radical expression is even, the radicand cannot be negative, so we must exclude any numbers which make the radicand negative from the domain.

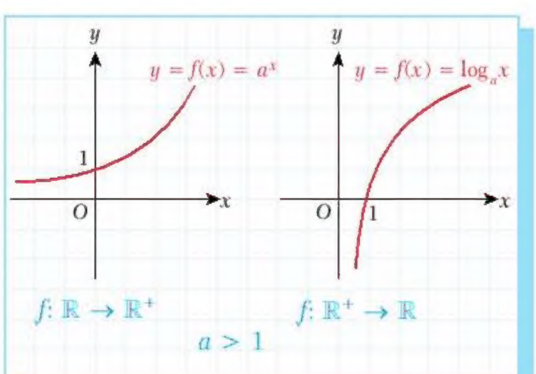
As we can see in the graph opposite, since the radicand is non-negative both the domain and range of $y = \sqrt[n]{x}$ are the set $\mathbb{R}^+ \cup \{0\}$.

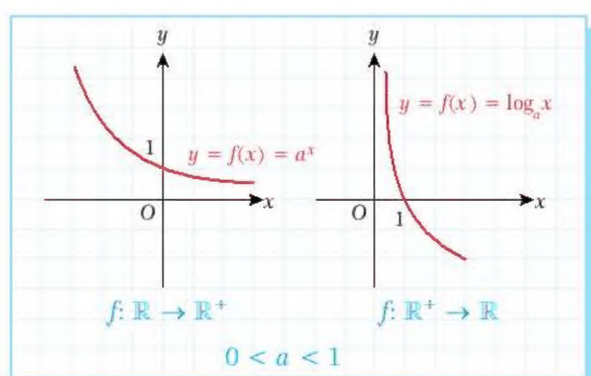


Type of function	Form	Domain	Examples
exponential function	$f(x) = a^x$ $(a \in \mathbb{R}^+ - \{1\})$	$\mathbb{R}$	$f(x) = 3^x$ $\quad D(f)$: $\mathbb{R}$ $f(x) = 5^{x^2+2x}$ $\quad D(f)$: $\mathbb{R}$
logarithmic function	$f(x) = \log_a g(x)$ $(a \in \mathbb{R}^+ - \{1\})$	$\mathbb{R} - \{x \mid g(x) \leq 0\}$	$f(x) = \log(x^2 - 4)$ $\quad D(f)$: $\mathbb{R} - [-2, 2]$ $f(x) = \ln(x^2 - 2x - 3)$ $\quad D(f)$: $\mathbb{R} - [-1, 3]$

An exponential function is defined for all real numbers but the logarithmic function is defined only for positive real numbers. Therefore we must exclude any numbers which make the function negative or zero from the domain of a logarithmic function.

Look at the following graphs of exponential and logarithmic functions:





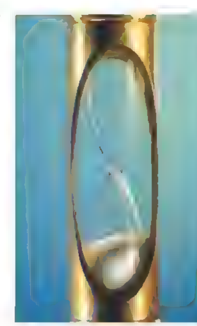
EXAMPLE 1 The following table shows some more examples of the domain and range of different functions. Write the missing values in the table.

Function	Domain	Range
$f(x) = x^2 + 2x + 1$	$\mathbb{R}$	$[0, +\infty)$
$f(x) = x^2 - 2x - 3$	$\mathbb{R}$	
$f(x) = x^3 - x^2 + x + 1$		$\mathbb{R}$
$f(x) = \dfrac{1}{x+1}$		$(-\infty, 0) \cup (0, +\infty)$
$f(x) = \dfrac{x^3 - 1}{x^2 - 5x + 6}$	$\mathbb{R} - \{2, 3\}$	
$f(x) = \sqrt{1 - x^2}$		$[0, 1]$
$f(x) = \log(x^2 + 5x)$	$(-\infty, -5) \cup (0, +\infty)$	

Solution The solution is left as an exercise for you.

EXAMPLE 2 Find the domain and range of the function $f(x) = \sqrt{x^2 + 5x + 6}$.

Solution We know that the radicand of a square root function cannot be negative. Let us look at the sign of the radicand $x^2 + 5x + 6$:



x		-3		-2	
$x^2 + 5x + 6$	$+$	0	$-$	0	$+$

The radicand is non-negative in the intervals $(-\infty, -3]$ and $[-2, +\infty)$.

Therefore the domain of the function is $(-\infty, -3] \cup [-2, +\infty)$.

As x increases, the value of y also increases without limit.

So the range is $[0, +\infty)$.

In conclusion, the domain of f is $(-\infty, -3] \cup [-2, +\infty)$ and the range is $[0, +\infty)$.

Note

If a function f is a sum or difference of different functions then the domain of f is the intersection of the domains of each function.

EXAMPLE 3 Find the domain of the function $f(x)=\dfrac{1}{\sqrt{x}}+\log(x^2+2x-8)$.

Solution We can think of this function as the sum of two functions: $g(x)=\dfrac{1}{\sqrt{x}}$ and $h(x) = \log(x^2 + 2x - 8)$.

$\dfrac{1}{\sqrt{x}}$ is defined when $0 < x < +\infty$.

The logarithmic function is defined in $\mathbb{R}^+$, so let us look at the sign of $x^2 + 2x - 8$:

x		-4		2	
$x^2 + 2x - 8$	$+$	0	$-$	0	$+$

We can see that $\log(x^2 + 2x - 8)$ is defined when $x < -4$ or $x > 2$.

The domain of f is the intersection of the domains of g and h, so $D(f) = (2, +\infty)$.

EXAMPLE 4 Find the domain of $f(x)=\sqrt{x+3}+\dfrac{5}{x^2-4}+\log_2(4x-3)$.

Solution Let us consider the domain of each separate function.

By solving the inequalities $\begin{cases} x+3\geq 0 \\ x^2-4\neq 0 \\ 4x-3>0 \end{cases}$, we get $\begin{cases} x\geq -3 \\ x\neq 2 \text{ and } x\neq -2 \\ x>\dfrac{3}{4}. \end{cases}$

So the domain of f is the intersection of these three intervals: $D(f)=\left(\dfrac{3}{4}, +\infty\right)-\{2\}$.

EXAMPLE 5 Find the image set of each function over the given interval.

a. $f(x) = 3x + 6$, $x \in [0, +\infty)$

b. $f(x) = x^2 - 2x + 8$, $x \in [-1, 2]$

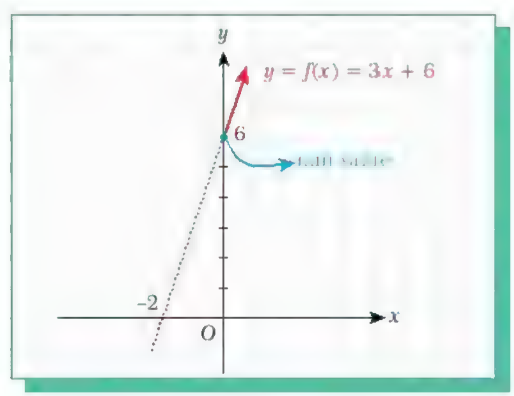
c. $f(x) = x^2 - 4x - 5$, $x \in [-1, 1]$

d. $f(x) = 2x - x^2$, $x \in [0, 3]$

Solution We can find each range by drawing the graph of the function over the given interval.

a. As we can see in the graph, $f(x)$ is an increasing function. The solid line shows the graph on the interval $x \in [0, +\infty)$. On this interval the minimum value of f is $f(0) = 6$, and the maximum value goes to infinity.

So the image set of f on this interval is $[6, +\infty)$.



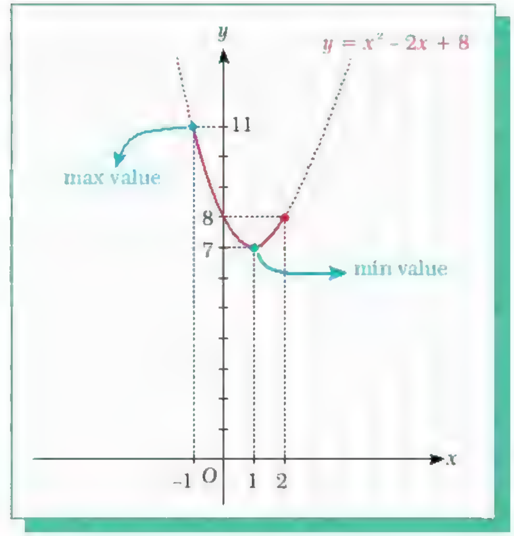
The image set I of a function $f: A \to B$ is the set of images of all the elements of A, so $I \subseteq B$.

b. The figure shows the graph of the function $y = f(x) = x^2 - 2x + 8$. The solid line shows the graph on the interval $x \in [-1, 2]$.

Since the extremum (vertex) of a function $f(x) = ax^2 + bx + c$ is $(-\frac{b}{2a}, f(-\frac{b}{2a}))$, the minimum value of the function on this interval is $f(-\frac{b}{2a}) = f(-\frac{-2}{2}) = f(1) = 7$.

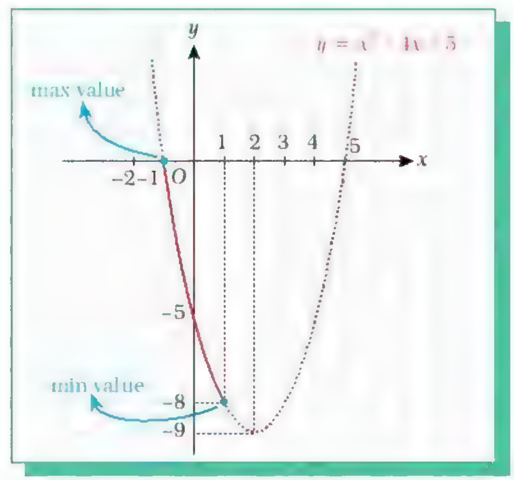
Its maximum value on this interval is $f(-1) = 11$.

So the image set of f on this interval is $[7, 11]$.



c. The figure shows the graph of $y = f(x) = x^2 - 4x - 5$. The solid line shows the graph on the interval $x \in [-1, 1]$.

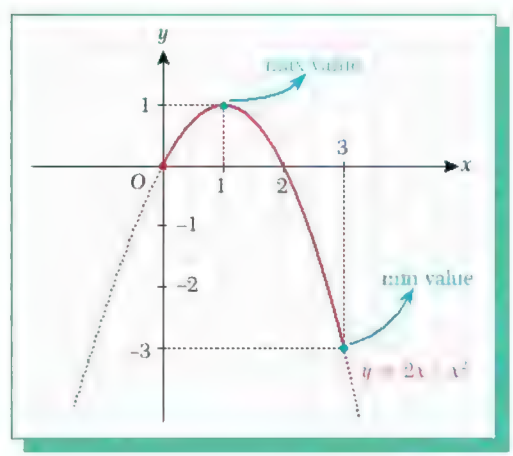
On this interval,
$\min f(x) = f(1) = -8$ and
$\max f(x) = f(-1) = 0$.
So the image set of f on this interval is $[-8, 0]$.



d. The solid line in the figure shows the graph of $y = f(x) = 2x - x^2$ on the interval $x \in [0, 3]$.

On this interval,
$\min f(x) = f(3) = -3$ and
$\max f(x) = f(1) = 1$.

So the image set of the function on this interval is $[-3, 1]$.



EXAMPLE 6 Find the range of each function for its largest domain.

a. $f(x) = 2\sin x - 3$ b. $f(x) = \sqrt{-x^2 + 4x + 5}$ c. $f(x) = \sqrt{5^x + 7}$ d. $f(x) = \dfrac{2}{3^x + 1}$

Solution a. The trigonometric function $f(x) = \sin x$ is defined from $\mathbb{R}$ to $[-1, 1]$.

For all $x \in \mathbb{R}$, $-1 \le \sin x \le 1$

$-2 \le 2\sin x \le 2$

$-2 - 3 \le 2\sin x - 3 \le 2 - 3$

$-5 \le 2\sin x - 3 \le -1$.

Hence the range of $f(x) = 2\sin x - 3$ is $[-5, -1]$.

b. Let us define $g(x) = -x^2 + 4x + 5$ and plot its graph (shown opposite).

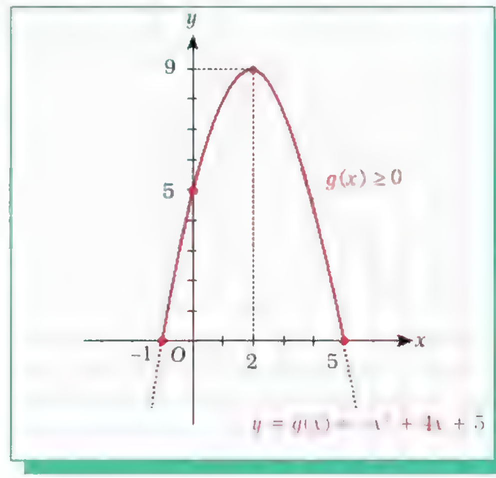
As we can see, the function is positive over the interval $x \in [-1, 5]$.

So $f(x) = \sqrt{-x^2 + 4x + 5}$ is defined on the interval $x \in [-1, 5]$. Also, on this interval the minimum value of $f(x)$ is 0 and the maximum value of $f(x)$ is $\sqrt{9}$, so we can write

$0 \le \sqrt{-x^2 + 4x + 5} \le \sqrt{9}$

$0 \le \sqrt{-x^2 + 4x + 5} \le 3$

$0 \le f(x) \le 3$. So the range of f is $[0, 3]$.



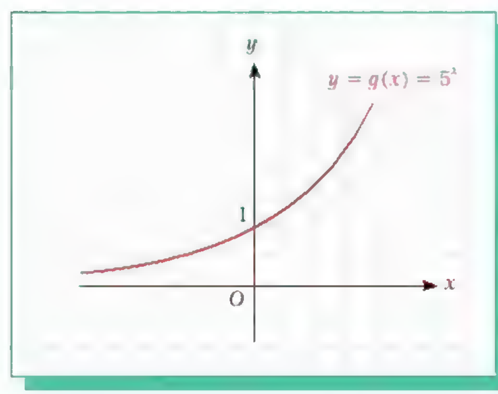
c. Since the exponential function $g(x) = 5^x$ is always positive, the domain of $f(x) = \sqrt{5^x + 7}$ is $\mathbb{R}$. As we can see in the graph opposite, the value of 5^x lies between 0 and ∞.

$0 < 5^x < \infty$

$7 < 5^x + 7 < \infty$

$\sqrt{7} < \sqrt{5^x + 7} < \infty$

So the range of f is $(\sqrt{7}, \infty)$.



d. Let us begin by considering the range of 3^x: $0 < 3^x < \infty$.

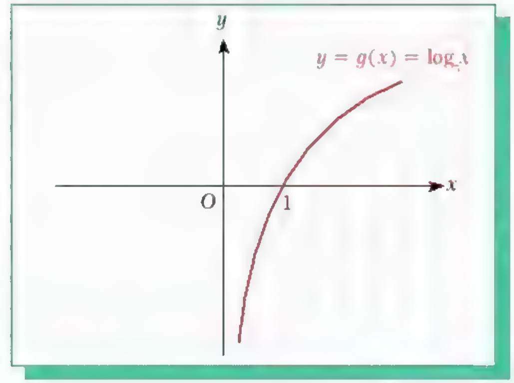
This gives us $1 < 3^x + 1 < +\infty$, and when we take the reciprocal of each side, the inequality signs are reversed.

So we get $\dfrac{1}{1} > \dfrac{1}{3^x + 1} > 0$, i.e. $2 > \dfrac{2}{3^x + 1} > 0$.

So the range of f is $(0, 2)$.

EXAMPLE 7 Find the domain of $f(x) = \log_5(\log_5 x)$.

Solution The logarithmic function is defined for all positive real numbers, so the function $g(x) = \log_5 x$ is defined from $\mathbb{R}^+$ to $\mathbb{R}$. Now we must specify on which interval $g(x)$ is non-positive (i.e. negative or zero).



As we can see in the graph, $g(x) = \log_5 x$ is non-positive for $0 < x \leq 1$, so $(0, 1]$ must be excluded from the domain of f. Therefore, the domain of f is $\mathbb{R}^+ - (0, 1]$.

EXAMPLE 8 Find the domain of $f(x) = \sqrt{|x-1| - |x+2|}$.

Solution The radicand $|x - 1| - |x + 2|$ must be non-negative, i.e.

$|x - 1| - |x + 2| \geq 0$.

This gives $|x - 1| \geq |x + 2|$. Taking the square of both sides gives us $x^2 - 2x + 1 \geq x^2 + 4x + 4$

$-6x \geq 3$

$x \leq -\frac{1}{2}$. So the domain of the function is $(-\infty, -\frac{1}{2}]$.

EXAMPLE 9 Find the domain of $f(x) = \sqrt{3 - \sqrt{12 - x^2}}$.

Solution We have the radicands $12 - x^2$ and $3 - \sqrt{12 - x^2}$, and both of them must be non-negative:

$12 - x^2 \geq 0$ and $3 - \sqrt{12 - x^2} \geq 0$

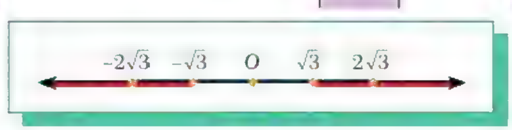
$$9 \geq 12 - x^2$$

$$x^2 - 3 \geq 0.$$

Let us solve each quadratic inequality by constructing its sign table:

x		$-2\sqrt{3}$		$2\sqrt{3}$	
$12 - x^2$	$-$	0	$+$	0	$-$

x		$-\sqrt{3}$		$\sqrt{3}$	
$x^2 - 3$	$+$	0	$-$	0	$+$



The domain of the combined function $f(x) = \sqrt{3 - \sqrt{12 - x^2}}$ is the intersection of these intervals: $D(f) = [-2\sqrt{3}, -\sqrt{3}] \cup [\sqrt{3}, 2\sqrt{3}]$.

EXAMPLE 10 Find the domain of $f(x)=\sqrt{\log\frac{5x-1}{x+2}}$.

Solution To find the domain we have to solve three inequalities:

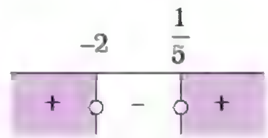
$x+2 \neq 0,$ (1)

$\frac{5x-1}{x+2} > 0$ (2)

and $\log\frac{5x-1}{x+2} \geq 0.$ (3)

(1) gives $x+2 \neq 0$, i.e. $x \neq -2$.

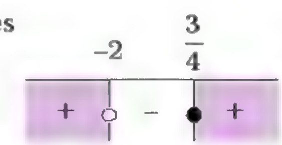
(2) gives $\frac{5x-1}{x+2} > 0,$



i.e. $x < -2$ or $x > \frac{1}{5}$.

(3) gives $\log\frac{5x-1}{x+2} \geq 0,\ \log\frac{5x-1}{x+2} \geq \log 1$ i.e. $\frac{5x-1}{x+2} \geq 1$. This gives

$\frac{5x-1}{x+2} - 1 \geq 0,\ \frac{5x-1-x-2}{x+2} \geq 0,\ \frac{4x-3}{x+2} \geq 0.$



The domain is the intersection of the three solution sets: $D(f) = (-\infty, -2) \cup [\frac{3}{4}, +\infty)$.

EXAMPLE 11 Find the domain of $f(x) = \arccos(3x+1)$.

Solution The trigonometric function $\cos x$ is defined from $\mathbb{R}$ to the closed interval $[-1, 1]$. Therefore its inverse $\arccos x$ will be defined from $[-1, 1]$ to $\mathbb{R}$. In other words,

$\cos x: \mathbb{R} \to [-1, 1],$

$\arccos x: [-1, 1] \to \mathbb{R}.$

This means that the value of $3x+1$ must lie between –1 and 1:

$-1 \leq 3x + 1 \leq 1$

$-2 \leq 3x \leq 0$

$-\frac{2}{3} \leq x \leq 0.$

So the domain of the function $f(x) = \arccos(3x+1)$ is $[-\frac{2}{3}, 0]$.

EXAMPLE 12 Find the domain and the range of $f(x) = \arcsin\sqrt{-x^2 + x}$.

Solution First let us find the domain of the function $g(x) = \sqrt{-x^2 + x}$.

The radicand must be non-negative, so

$-x^2 + x \geq 0$,

$-x(x - 1) \geq 0$.

x		0		1	
$-x(x-1)$	$-$	$\bullet$	$+$	$\bullet$	$-$

This is true for $0 \leq x \leq 1$, so the domain of $g(x)$ is $[0, 1]$.

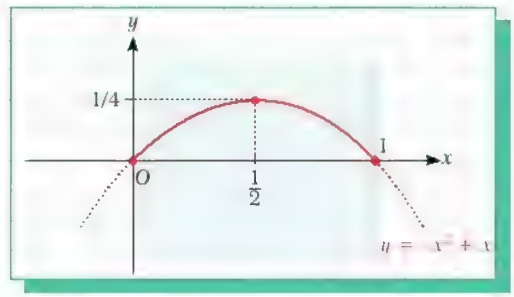
Now let us find the range of $g(x) = \sqrt{-x^2 + x}$.

Let x_1 and x_2 be the roots of the equation $-x^2 + x = 0$, i.e. $x_1 = 0$ and $x_2 = 1$.

For $x \in [0, 1]$ the expression $-x^2 + x$ reaches its maximum value at $x = \dfrac{x_1 + x_2}{2} = \dfrac{0+1}{2} = \dfrac{1}{2}$

which is $(\frac{1}{2}, \frac{1}{4})$. It reaches its minimum value at $x = 0$.

So $g(\frac{1}{2}) = \sqrt{-(\frac{1}{2})^2 + \frac{1}{2}} = \sqrt{-\frac{1}{4} + \frac{1}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$ is the maximum value of $g(x)$.



Similarly, $g(0) = g(1) = 0$ is the minimum value of $g(x)$.

Hence the range of $g(x) = \sqrt{-x^2 + x}$ is $[0, \frac{1}{2}]$.

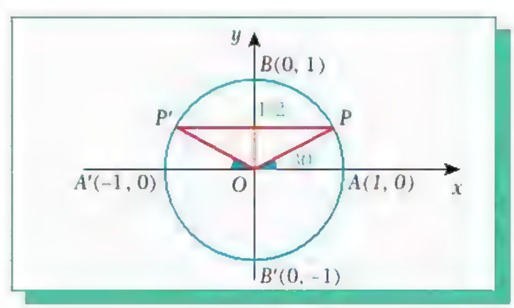
Since the function $\arcsin x$ is defined on the interval $[0, \frac{1}{2}]$, the domain of $f(x) = \arcsin\sqrt{-x^2 + x}$ is also the interval $[0, 1]$.

Now we can find the range of $f(x) = \arcsin\sqrt{-x^2 + x} = \arcsin(g(x))$.

We have found that $0 \leq \sqrt{-x^2 + x} \leq \frac{1}{2}$. So the range of the function is the set of angles whose sine value is between 0 and $\frac{1}{2}$. As we can see in the figure:

for $0° \leq \theta \leq 30°$, $\quad 0 \leq \sin\theta \leq \frac{1}{2}$ and

for $150° \leq \theta \leq \pi$, $\quad 0 \leq \sin\theta \leq \frac{1}{2}$.



So the range of $f(x) = \arcsin\sqrt{-x^2 + x}$ is the set $[0, \frac{\pi}{6}] \cup [\frac{5\pi}{6}, \pi]$, and we can write

$f : [0, 1] \to [0, \frac{\pi}{6}] \cup [\frac{5\pi}{6}, \pi]$.

Check Yourself 1

1. Find the domain of each function.

a. $f(x)=\dfrac{x}{x^2+2}$ b. $f(x)=\sqrt[3]{2x+1}+\sqrt{x^2-1}$ c. $f(x)=2x-\sqrt[4]{\dfrac{x+1}{x-1}}$

d. $f(x)=\sqrt{2x-1-\sqrt{3x+1}}$ e. $f(x)=3^{\frac{1}{x-2}}$ f. $f(x)=2^{\log_3\left(\frac{x+2}{x^2-4}\right)}$

g. $f(x)=\sqrt{3-\log_2 x}$ h. $f(x)=\sqrt{\log_2\left(\dfrac{x+2}{2x+2}\right)}$ i. $f(x)=\log_{\frac{1}{2}}(\log_2(2x+1))$

j. $f(x) = \arccos(2x-3)$ k. $f(x)=\arcsin\left(\dfrac{2x+1}{x+1}\right)$ l. $f(x)=\dfrac{3x+1}{\log(x^2+x+1)-1}$

m. $f(x)=\dfrac{\sqrt{2-\cos x}}{2+\sin x}$ n. $f(x)=\sqrt{\tan x},\ \ x\in[0,\ \pi)$

2. Find the image of each function over the given interval.

a. $f(x) = 2x + 1,\ x \in [1, 5)$ b. $f(x) = x^2 - 4x - 5,\ x \in [1, 3)$

c. $f(x) = \cos x + \sin x,\ x \in [0, 2\pi)$ d. $f(x) = \log_3(x^2 - 2x - 8),\ x \in [6, 7]$

e. $f(x) = 2^{x+1},\ x \in [-1, 3)$ f. $f(x)=\sqrt{x^2+2},\ \ x\in(\sqrt{2},\sqrt{7}]$

3. Find the range of each function for its largest domain.

a. $f(x)=\sqrt{-x^2+7x-12}$ b. $f(x) = -3\cos x + 1$ c. $f(x)=\dfrac{5}{2^x+3}$

Answers

1. a. $\mathbb{R}$ b. $\mathbb{R}-(-1, 1)$ c. $\mathbb{R}-(-1, 1]$ d. $[\frac{7}{4},\ \infty)$ e. $\mathbb{R}-\{2\}$ f. $(2, \infty)$ g. $(0, 8]$ h. $(-1, 0]$

i. $(0, \infty)$ j. $[1, 2]$ k. $[-\frac{2}{3},\ 0]$ l. $\mathbb{R}-\{\frac{-1+\sqrt{37}}{2},\ \frac{-1-\sqrt{37}}{2}\}$ m. $\mathbb{R}$ n. $[0,\ \frac{\pi}{2})$

2. a. $[3, 11)$ b. $[-9, -8]$ c. $[-\sqrt{2}, \sqrt{2}]$ d. $[4\log_3 2,\ 3]$ e. $[1, 16)$ f. $(2, 3]$

3. a. $[0,\ \frac{1}{2}]$ b. $[-2, 4]$ c. $(0,\ \frac{5}{3})$

B. COMPOSITE FUNCTION

Definition **composite function**

A function that is formed by the composition of two or more elementary functions is called a composite function.

$$x \longrightarrow \overset{f}{\boxed{\quad}} \xrightarrow{f(x)} \overset{g}{\boxed{\quad}} \longrightarrow g(f(x))$$

The function $g(f(x))$ is called the composite of **g** with f.

The composite function $g(f(x))$ is also sometimes written as $(g \circ f)(x)$. The composition of functions is associative but not commutative, i.e. $(f \circ (g \circ h))(x) = ((f \circ g) \circ h)$ but $(f \circ g)(x) \neq (g \circ f)(x)$.

EXAMPLE 13 Given $f(x) = x^2$, $g(x) = \sin x$ and $h(x) = 2x + 5$, find

a. $f(g(x))$. b. $g(f(x))$. c. $f(h(g(x)))$.

Solution a. $f(g(x)) = f(\sin x) = \sin^2 x$

b. $g(f(x)) = g(x^2) = \sin x^2$

c. $f(h(g(x))) = f(h(\sin x)) = f(2\sin x + 5) = (2\sin x + 5)^2$

EXAMPLE 14 Write each function as a composite function.

a. $f(x) = (3x - 5)^3$ b. $f(x) = 2^{5x^2-7}$ c. $f(x) = \sin^2(\log_3{}^7(x^2 - 1))$

Solution a. Let us choose $g(x) = 3x - 5$ and $h(x) = x^3$, then $f(x) = (3x - 5)^3 = h(g(x))$.

b. Let us choose $g(x) = x^2$, $h(x) = 5x - 7$ and $t(x) = 2^x$, then $f(x) = 2^{5x^2-7} = t(h(g(x)))$.

c. Let us choose $g(x) = x^2 - 1$, $h(x) = \log_3 x$, $t(x) = x^7$ and $u(x) = \sin^2 x$ so $f(x) = \sin^2(\log_3{}^7(x^2 - 1)) = u(t(h(g(x))))$.

Notice that these are not the only possible solutions to the question. We could have chosen different elementary functions and still achieved the same function. For example, for the function $f(x) = (3x - 5)^3$, we could have chosen $g(x) = 3x$, $h(x) = x - 5$ and $t(x) = x^3$ to get $f(x) = (3x - 5)^3 = t(h(g(x)))$.

Check Yourself 2

Write each function as a composite function.

1. $f(x) = \left(\dfrac{2x-1}{7}\right)^3$ 2. $f(x) = \sqrt{3 + \log_2 x}$ 3. $f(x) = \sin^7\left(\dfrac{x+5}{x^2+1}\right)$

C. INVERSE OF A FUNCTION

Recall the definition of inverse function: if the function $f: D \to R$ is both a one-to-one function and an onto function then the function $f^{-1}: R \to D$ is called the inverse of f.

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$

A function $f: A \to B$ is a one-to-one function if for each $x_1 \neq x_2$ in A, $f(x_1) \neq f(x_2)$.

A function $f: A \to B$ is an onto function if for any $y \in B$ there exists an $x \in A$ such that $f(x) = y$.

To find the inverse of a given function $y = f(x)$ it is enough to find x in terms of the variable y.

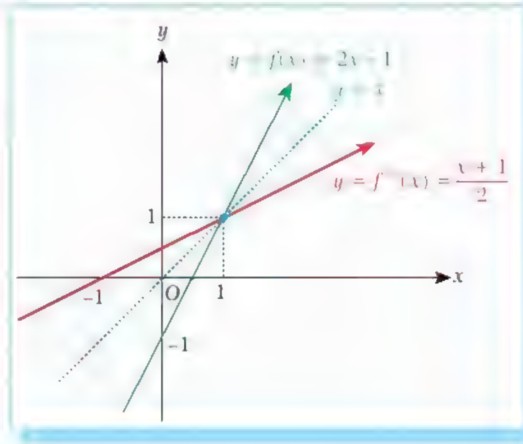
For example, let us find the inverse of the polynomial function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 1$:

Write $y = f(x)$: $y = 2x - 1$.

Express x in terms of y: $x = \dfrac{y+1}{2} = f^{-1}(y)$.

Finally, express the inverse function in terms of the variable x: $f^{-1}(x) = \dfrac{x+1}{2}$.

This is the inverse of $f(x) = 2x - 1$.



Recall that the graph of a function and the graph of its inverse are symmetric with respect to the line $y = x$.

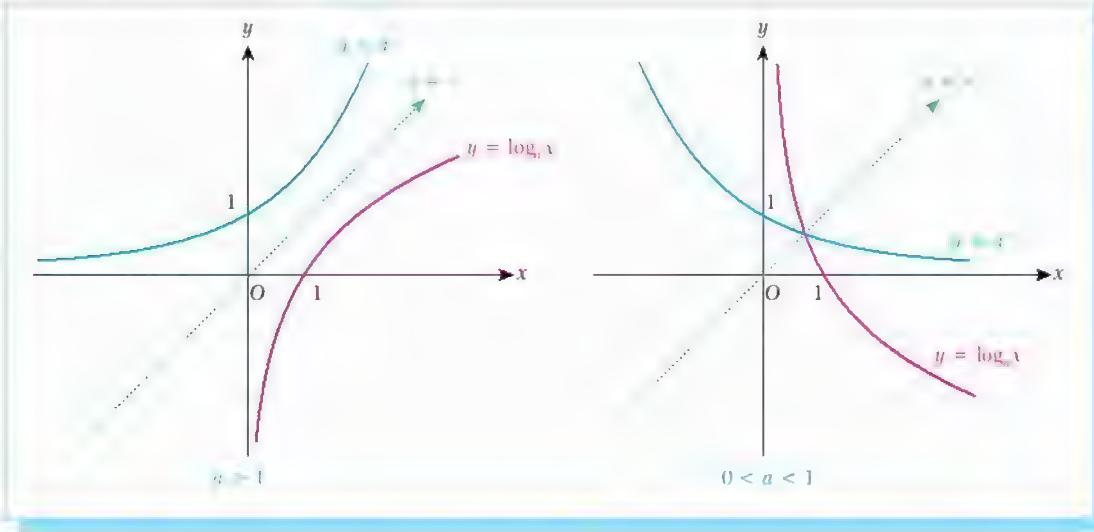
Let us recall the inverse of some common types of function:

Function	Form	Inverse
linear function	$f(x) = ax + b$	$f^{-1}(x) = \dfrac{x-b}{a}$
rational function	$f(x) = \dfrac{ax+b}{cx+d}$	$f^{-1}(x) = \dfrac{-dx+b}{cx-a}$

Remember that the exponential function and the logarithmic function are inverse of each other:

Function	Form	Inverse
exponential function	$f(x) = a^x$ $(a \in \mathbb{R}^+ - \{1\})$	$f^{-1}(x) = \log_a x$
logarithmic function	$f(x) = \log_a x$ $(a \in \mathbb{R}^+ - \{1\})$	$f^{-1}(x) = a^x$

As we can see in the figure below, the graphs of the exponential and logarithmic functions are symmetric with respect to the line $y = x$.



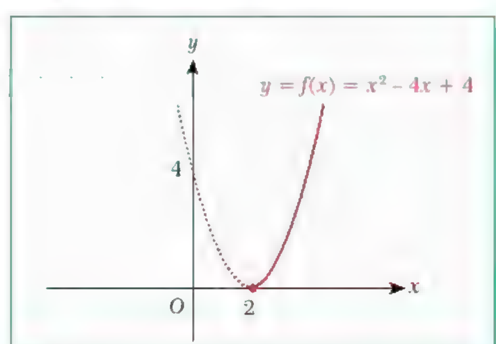
Look at some more examples of inverse functions:

Function	Inverse
$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \dfrac{3x + 1}{2}$	$f^{-1}: \mathbb{R} \to \mathbb{R}, \quad f^{-1}(x) = \dfrac{2x - 1}{3}$
$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = 2 - 3x$	$f^{-1}: \mathbb{R} \to \mathbb{R}, \quad f^{-1}(x) = \dfrac{-x + 2}{3}$
$f: \mathbb{R} \to \mathbb{R}^+, \quad f(x) = 3^x$	$f^{-1}: \mathbb{R}^+ \to \mathbb{R}, \quad f^{-1}(x) = \log_3 x$
$f: \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}, \quad f(x) = \dfrac{2x + 1}{x - 1}$	$f^{-1}: \mathbb{R} - \{2\} \to \mathbb{R} - \{1\}, \quad f^{-1}(x) = \dfrac{x + 1}{x - 2}$

EXAMPLE 15 Find the inverse of the function $f: [2, \infty) \to [0, \infty)$, $f(x) = x^2 - 4x + 4$.

Solution In the given domain and range, $f(x)$ is both one-to-one and onto, so its inverse is a function. We can find this inverse function as follows:

$$\begin{aligned} x^2 - 4x + 4 &= y \\ (x - 2)^2 &= y \\ x - 2 &= \sqrt{y} \\ x &= \sqrt{y} + 2 = f^{-1}(y). \end{aligned}$$



So the inverse is

$f^{-1}: [0, \infty) \to [2, \infty)$, $f^{-1}(x) = \sqrt{x} + 2$.

EXAMPLE 16 Find the inverse of the function $f: \mathbb{R} \to (2, \infty)$, $f(x) = 10^{x-1} + 2$.

Solution

$$f(x) = 10^{x-1} + 2 = y$$
$$10^{x-1} = y - 2$$
$$x - 1 = \log_{10}(y - 2)$$
$$x = \log_{10}(y - 2) + 1 = f^{-1}(y)$$

So the inverse is $f^{-1}: (2, \infty) \to \mathbb{R}$, $f^{-1}(x) = \log_{10}(x - 2) + 1$ or $f^{-1}(x) = \log(10x - 20)$.

EXAMPLE 17 Find the inverse of the function $f: (-4, \infty) \to \mathbb{R}$, $f(x) = -1 + 3\log_2(x + 4)$.

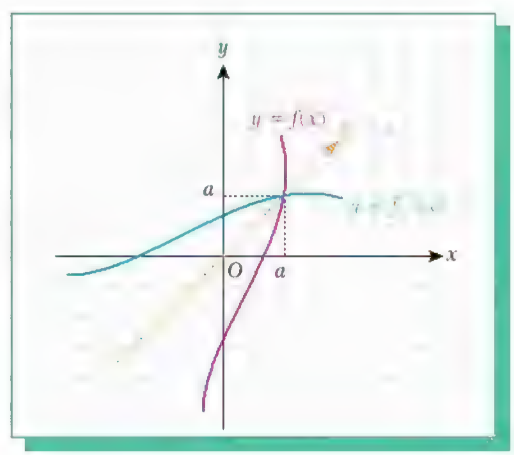
Solution

$$f(x) = -1 + 3\log_2(x + 4) = y$$
$$\log_2(x + 4) = \frac{y+1}{3}$$
$$x + 4 = 2^{(\frac{y+1}{3})}$$
$$x = 2^{(\frac{y+1}{3})} - 4 = f^{-1}(y)$$

So the inverse is $f^{-1}: \mathbb{R} \to (-4, \infty)$, $f^{-1}(x) = 2^{(\frac{x+1}{3})} - 4$.

EXAMPLE 18 The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x^2 + 4x - 1$ is given. Find the real number a which satisfies the equation $f(a) = f^{-1}(a)$.

Solution To solve the problem we have to find the intersection of the graphs of f and f^{-1}. However, we know that the graph of a function and the graph of its inverse are symmetric with respect to the line $y = x$. In other words, the intersection of the two graphs will be on this line. At the intersection point, therefore, $y = f(x) = x$, and so $f(a) = a = f^{-1}(a)$, as shown at the right.



If $f(a) = a$ then $a^3 - 3a^2 + 4a - 1 = a$

$$a^3 - 3a^2 + 3a - 1 = 0$$
$$(a - 1)^3 = 0$$
$$a = 1.$$

Check Yourself 3

1. Find the inverse of each function.

   a. $f: \mathbb{R} \to \mathbb{R},\ f(x) = 3 - \dfrac{x}{2}$

   b. $f: \mathbb{R} - \{1\} \to \mathbb{R} - \left\{\dfrac{5}{2}\right\},\ f(x) = \dfrac{5x-2}{2x-2}$

   c. $f: \mathbb{R} \to \mathbb{R}^+,\ f(x) = 5^{2x-1}$

   d. $f: \mathbb{R}^+ \to \mathbb{R},\ f(x) = \log_3(2x + 5)$

   e. $f: \mathbb{R} \to \mathbb{R},\ f(x) = x^3 - 3x^2 + 3x$

   f. $f: \mathbb{R} \to (1, \infty),\ f(x) = 3^{x-1} + 1$

   g. $f: \mathbb{R} \to (1, \infty),\ f(x) = 1 + \log(3^x + 1)$

   h. $f: \mathbb{R} - \{1\} \to (1, \infty) - \{11\},\ f(x) = 10^{\frac{x+1}{x-1}} + 1$

2. For each function, find the real number a which satisfies the equation $f(a) = f^{-1}(a)$.

   a. $f(x) = 3x + 1$ b. $f(x) = 8x^3 - 12x^2 + 7x - 1$

Answers

1. a. $6 - 2x$ b. $\dfrac{2x-2}{2x-5}$ c. $\log_5 \sqrt{x} + \dfrac{1}{2}$ d. $\dfrac{3^x - 5}{2}$ e. $\sqrt[3]{x-1} + 1$ f. $\log_3(x - 1) + 1$

g. $\log_3(10^{x-1} - 1)$ h. $\dfrac{2}{\log \dfrac{x-1}{10}} + 1$ 2. a. $-\dfrac{1}{2}$ b. $\dfrac{1}{2}$

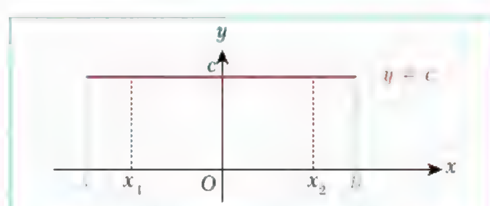
D. CONSTANT, INCREASING AND DECREASING FUNCTIONS

Let $f: D \to R$ be a function and let $I \subset D$ and $x_1, x_2 \in I$ and such that $x_1 < x_2$.

1. If $f(x_1) = f(x_2)$ for all $x_1, x_2 \in I$ then f is called a constant function on the interval I.

   We write a constant function as $f(x) = c$ $(c \in \mathbb{R})$.

   If f is a constant function,
   $x_1 < x_2 \Leftrightarrow f(x_1) = f(x_2) = c$.



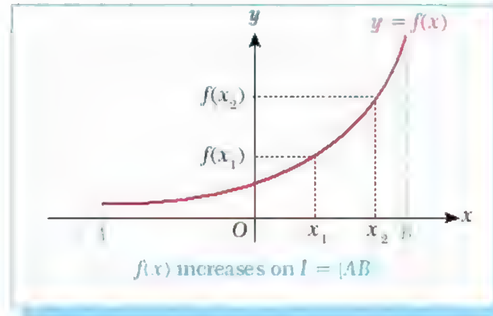
$f(x)$ is constant on $I = [AB]$

2. If $f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$ then f is called an increasing function on the interval I.

If f is an increasing function,

$x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$.

> If $x_1 < x_2$ and $f(x_1) \le f(x_2)$ then f is called a non-decreasing function.



$f(x)$ increases on $I = [AB]$

Note

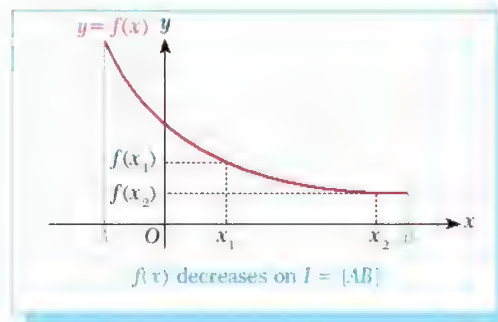
The increasing function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$ is called the identity function.

3. If $f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$ then f is called a decreasing function on the interval I.

If f is a decreasing function,

$x_1 < x_2 \Leftrightarrow f(x_1) > f(x_2)$.

> If $x_1 < x_2$ and $f(x_1) \ge f(x_2)$ then f is called a non-increasing function.



$f(x)$ decreases on $I = [AB]$

EXAMPLE 19 Given that $f(x) = (5 - a)x^2 + (b + 2)x - 3$ is a constant function, find a and b.

Solution Since f is a constant function, the coefficients of x^2 and x must be zero:

$5 - a = 0$ and $b + 2 = 0$. So $a = 5$ and $b = -2$.

EXAMPLE 20 Show that

a. $f: [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x} + x^3$ is an increasing function.

b. $f: \mathbb{R} - \{0\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x}$ is a decreasing function.

Solution a. Let $x_1, x_2 \in [0, \infty)$ such that $x_1 < x_2$.

For all $x_1 < x_2$, $\sqrt{x_1} < \sqrt{x_2}$ and $x_1^3 < x_2^3$. So $\sqrt{x_1} + x_1^3 < \sqrt{x_2} + x_2^3$ and $f(x_1) < f(x_2)$.

So f is an increasing function on $[0, \infty)$.

b. First let $x_1, x_2 \in (0, \infty)$ such that $x_1 < x_2$. For all $x_1 < x_2$, $\dfrac{1}{x_1} > \dfrac{1}{x_2}$ and so $f(x_1) > f(x_2)$.

Now let $x_1, x_2 \in (-\infty, 0)$ such that $x_1 < x_2$. For all $x_1 < x_2$, $\dfrac{1}{x_1} > \dfrac{1}{x_2}$ and so $f(x_1) > f(x_2)$.

In both cases f is decreasing, so f is a decreasing function on $\mathbb{R} - \{0\}$.

EXAMPLE 21

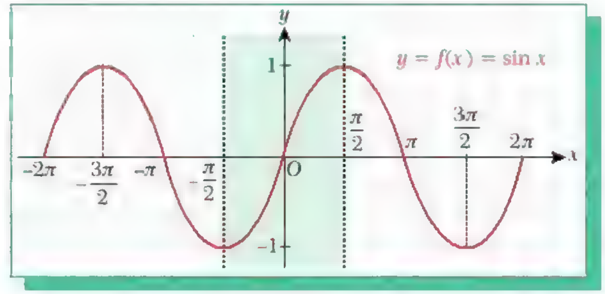
Determine whether each function increases or decreases on the given interval.

a. $y = \sin x,\ x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ b. $y = 0.5^x - 3x,\ x \in \mathbb{R}$ c. $y = \ln x + x^2,\ x \in \mathbb{R}^+$

Solution

a. $y = \sin x$ is a periodic function. Its graph is shown in the figure opposite.

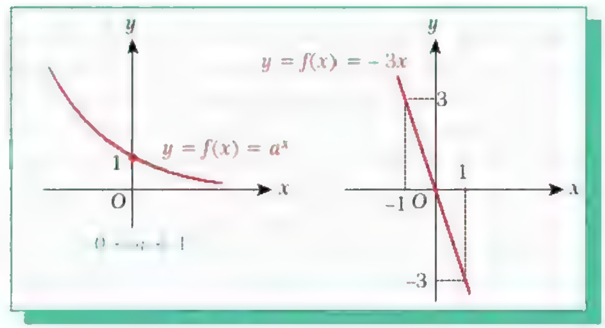
We can see that on the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, $y = \sin x$ increases.



> Let $t \in \mathbb{R}$. If $f(x + t) = f(x)$ then f is called a periodic function.

b. $y = \left(\dfrac{1}{2}\right)^x + (-3x)$

Recall that the exponential function $f(x) = a^x$ increases when $a > 1$ and decreases when $0 < a < 1$ as shown in the figure opposite.



The exponential function $\left(\dfrac{1}{2}\right)^x$ is therefore a decreasing function because $a = \dfrac{1}{2} < 1$.

> The sum of two increasing functions is also an increasing function. The sum of two decreasing functions is also a decreasing function.

We can easily see from the graph of $y = -3x$ that this is also a decreasing function. The sum of two decreasing functions is also a decreasing function, so f is a decreasing function.

c. $y = \ln x + x^2$

$\ln x$ is an increasing function because $\ln x = \log_e x$ and $e \cong 2.71 > 1$.

x^2 is also an increasing function in $\mathbb{R}^+$, so $y = \ln x + x^2$ is increasing function.

EXAMPLE 22

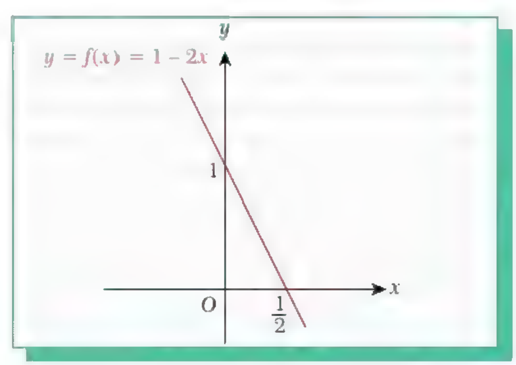
Find the interval(s) on which each function decreases and/or increases.

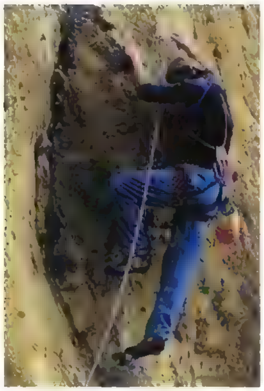
a. $y = 1 - 2x$ b. $y = x^2 - 3x$ c. $y = \dfrac{1}{x^2}$ d. $y = \cos x$

Solution

We can draw a graph of each function to determine the intervals.

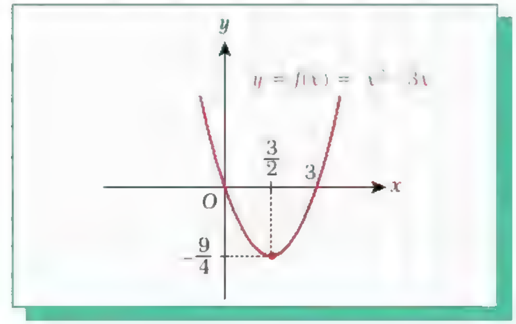
a. We can see from the graph that $f(x) = 1 - 2x$ decreases on $(-\infty, \infty)$.





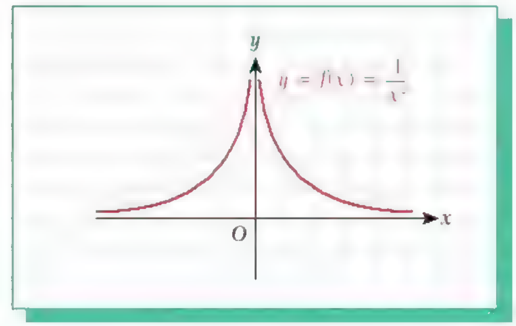
b. f decreases on the interval $\left(-\infty, \frac{3}{2}\right]$.

f increases on the interval $\left[\frac{3}{2}, +\infty\right)$.



c. f increases on the interval $(-\infty, 0)$.

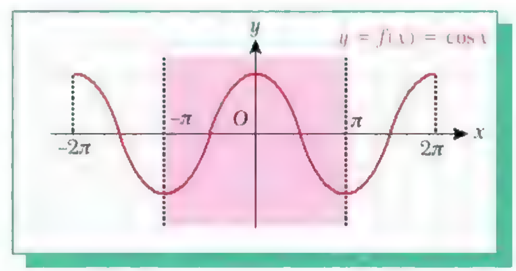
f decreases on the interval $(0, +\infty)$.



d. $f(x) = \cos x$ is a periodic function, so let us consider its value in the interval $[-\pi, \pi]$.

f increases on the interval $[-\pi, 0]$.

f decreases on the interval $[0, \pi]$.



EXAMPLE 23 Find the value $a + b$ if $f(x) = \dfrac{(a-2)x^2 + bx + 4}{3x + 2}$ is a constant function.

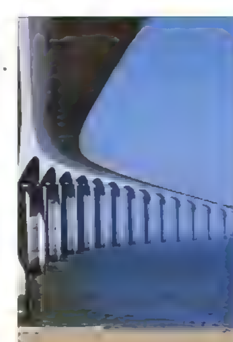
Solution Since f is constant, we can write $f(x) = \dfrac{(a-2)x^2 + bx + 4}{3x + 2} = k \quad (k \in \mathbb{R})$.

This gives $(a-2)x^2 + bx + 4 = 3kx + 2k$.

By the equality of polynomials, we can write

$$(a-2) = 0$$
$$b = 3k$$
$$4 = 2k, \text{ which gives}$$

$a = 2$, $k = 2$, $b = 6$. So $a + b = 8$.



EXAMPLE 24 The function $f(x)=\dfrac{(a-1)x^2+(b+5)x}{2x+1}$ is an identity function. Find a and b.

Solution Since f is an identity function, $f(x) = x$. So

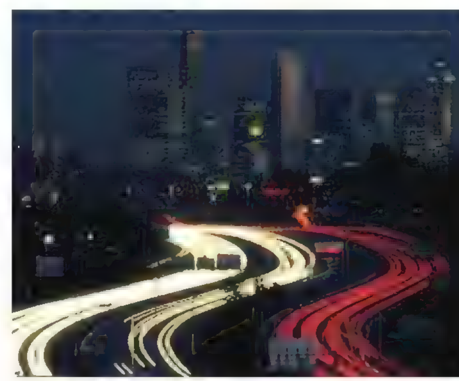
$$\frac{(a-1)x^2+(b+5)x}{2x+1}=x, \text{ i.e.}$$

$$(a-1)x^2+(b+5)x=2x^2+x.$$

By the equality of polynomials, $a - 1 = 2$ and $b + 5 = 1$. So $a = 3$ and $b = -4$.

Check Yourself 4

1. $f(x)=\dfrac{3mx+1}{6x+5}$ is a constant function. Find m.
2. $f(x) = (m - 2)x + n + 5$ is an identity function. Find $m + n$.
3. Decide whether each function increases or decreases on the given interval.
   a. $f(x) = 2x + 1,\ x \in \mathbb{R}$
   b. $f(x) = 1 - x,\ x \in \mathbb{R}$
   c. $f(x) = -x^2 - 8x + 1,\ x \in (-4, \infty)$
   d. $f(x) = -x^2 - 2x - 1,\ x \in (-\infty, 1)$
   e. $f(x) = x^3 + 1,\ x \in \mathbb{R}$



Answers

1. $\dfrac{2}{5}$ 2. -2

E. EVEN AND ODD FUNCTIONS

Definition even function, odd function

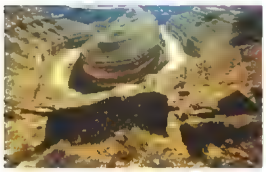
Let $f: D \to R$ be a function.

1. If $f(-x) = f(x)$ for all $x \in D$ then f is called an even function.
2. If $f(-x) = -f(x)$ for all $x \in D$ then f is called an odd function.

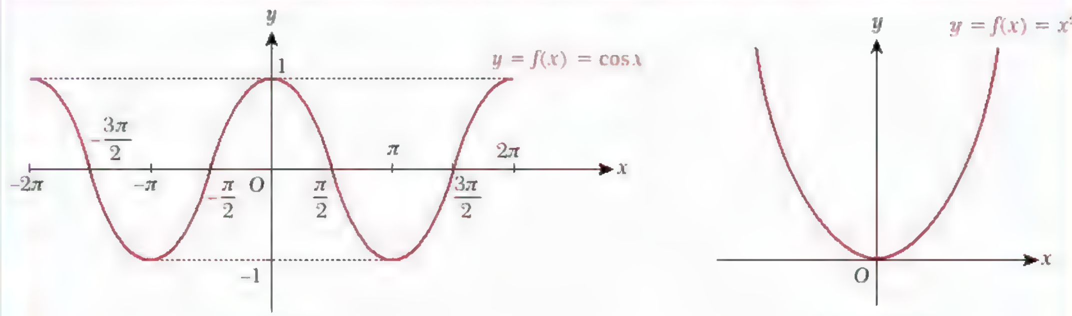
For example, the cosine function is an even function because $\cos(-x) = \cos x$.

Similarly, the sine, tangent and cotangent functions are odd functions because $\sin(-x) = -\sin x$, $\tan(-x) = -\tan x$ and $\cot(-x) = -\cot x$.

Not all functions are even or odd. For example, $f(x) = x + 1$ is neither even nor odd.



The graph of an even function is symmetric with respect to the y-axis.



$f(-x) = f(x)$ $\quad$ $\cos(-x) = \cos(x)$

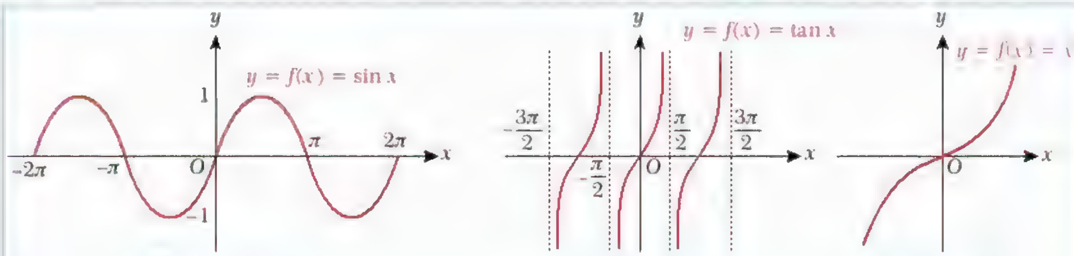
Even functions

$f(-x) = f(x)$ $\quad$ $(-x)^2 = (x)^2$

The following rules help us to calculate the parity (even or odd) of the sum and product of even functions (E) and odd functions (O):

$E \pm E = E$
$O \pm O = O$
$E \pm O =$ neither E nor O
$E \cdot E = E$
$E \cdot O = O$
$O \cdot O = E$.

The graph of an odd function is symmetric with respect to the origin.



$f(-x) = -f(x)$ $\quad$ $\sin(-x) = -\sin(x)$

$f(-x) = -f(x)$ $\quad$ $\tan(-x) = -\tan(x)$

$f(-x) = -f(x)$ $\quad$ $(-x)^3 = -x^3$

Odd functions

EXAMPLE 25 Determine whether each function is even, odd, or neither even nor odd.

a. $f(x) = x^2 + 3x + 2$ $\quad$ b. $f(x) = 7\tan x + x^3$ $\quad$ c. $f(x) = \dfrac{x^2 - 2}{x^6 + x^8}$ $\quad$ d. $f(x) = 3^x + 3^{-x}$

e. $f(x) = \sqrt{x^2 - 6x + 9} + \sqrt{x^2 + 6x + 9}$ $\quad$ f. $f(x) = |x - 3| + |x + 4|$ $\quad$ g. $f(x) = -3x^2 + 2|x| - 5$

Solution Let us find $f(-x)$ and compare it with $f(x)$ in each case.

a. $f(-x) = x^2 - 3x + 2$, so f is neither even nor odd.

b. $f(-x) = -7\tan x - x^3 = -(7\tan x + x^3) = -f(x)$, so f is odd.

c. $f(-x) = \dfrac{x^2 - 2}{x^6 + x^8} = f(x)$, so f is even.

d. $f(-x) = 3^{-x} + 3^x = 3^x + 3^{-x} = f(x)$, so f is even.

e. $f(-x) = \sqrt{x^2 + 6x + 9} + \sqrt{x^2 - 6x + 9} = f(x)$, so f is even.

f. $f(-x) = |-x - 3| + |-x + 4|$, so f is neither even nor odd.

g. $f(-x) = -3x^2 + 2|x| - 5 = f(x)$, so f is even.

EXAMPLE 26 $f: \mathbb{R} \to \mathbb{R}$, $f(x)$ is an odd function such that $f(-2) = k + 5$ and $f(2) = 2k + 3$. Find k.

Solution Since f is an odd function, $f(-x) = -f(x)$ and $f(-2) = -f(2)$.

So $k + 5 = -(2k + 3)$

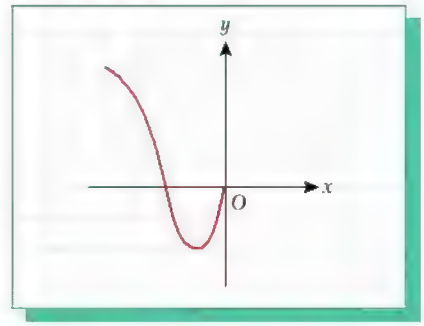
$k + 5 = -2k - 3$

$3k = -8$

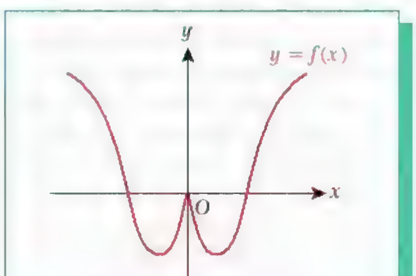
$k = -\frac{8}{3}$.

EXAMPLE 27 Complete the graph of the function if it is

a. even. b. odd.

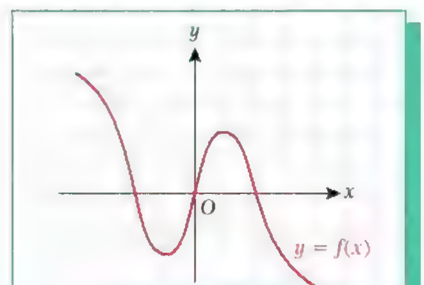


Solution a. The graph of an even function is symmetric with respect to the y-axis.



$f(x)$ is an even function

b. The graph of an odd function is symmetric with respect to the origin.



$f(x)$ is an odd function

Check Yourself 5

1. Determine whether each function is even, odd, or neither even nor odd.

a. $f(x) = |x| + \cos x$ b. $f(x) = x^3 + \sin x$ c. $f(x) = x^4 + x^2 + 1$

d. $f(x) = \cos x^4 - x^3 \sin x$ e. $f(x) = \dfrac{x}{\cos(x^3)}$ f. $f(x) = 2^{\frac{\sin x + \tan x}{x^3}}$

2. f is an odd function and g is an even function. $f(-2) + g(1) = 8$ and $g(-1) + f(2) = 6$ are given. Find $f(-2)$ and $g(-1)$.

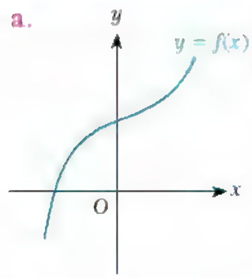
Answers

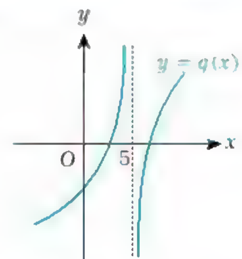
1. a. even b. odd c. even d. even e. odd f. even 2. $f(-2) = 1$, $g(-1) = 7$

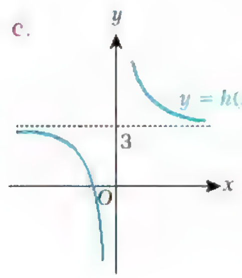
EXERCISES 1.1

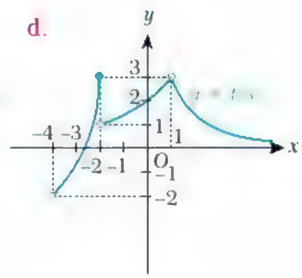
A. Domain and Range of a Function

1. State the domain and range of each function.

a.


b.


c.


d.


2. Find the domain of each function.

a. $f(x)=\sqrt[5]{\dfrac{x+3}{x-3}}$

b. $f(x)=\log_3(x^2+5x+6)$

c. $f(x)=5^{\sqrt{\frac{1}{x+1}}}$

d. $f(x)=\sqrt{9-|x^2-4|}$

e. $f(x)=\dfrac{\sqrt{x^2-3x-4}}{x^2-1}$

f. $f(x)=\sqrt{3x^2+1+\sqrt{\dfrac{2x}{x-5}}}$

g. $f(x)=\ln(x^2-7x+10)+\sqrt{x^2-4}$

h. $f(x)=\sqrt{\dfrac{1-2\sin x}{2}},\ x\in[0,2\pi]$

i. $f(x)=\arccos\dfrac{2-x}{x+1}$

j. $f(x)=\arcsin 2^x$

k. $f(x)=\sqrt{\log_{\frac{1}{2}}\left(\dfrac{x}{x^2-4}\right)}$

l. $f(x)=\log_{2x-5}(x^2-3x-10)$

m. $f(x)=\log(\log^2 x-3\log x-10)$

n. $f(x)=\sqrt{\cos^2 x-\cos x},\ \ x\in(0,2\pi)$

3. Find the range of each function over the given interval.

a. $f(x)=1-3x,\quad x\in[-2,4)$

b. $f(x)=x^2-2x-3,\quad x\in(2,4]$

c. $f(x)=-x^2+4x+5,\quad x\in[0,1)$

d. $f(x)=\log_{\frac{1}{2}}(3x+1),\quad x\in(1,5)$

e. $f(x)=3^{2x+5},\quad x\in(-1,1)$

f. $f(x)=\left(\dfrac{1}{2}\right)^{x+2},\quad x\in[-2,2)$

4. Find the range of each function for its largest domain.

a. $f(x)=\sqrt{-x^2+4}$

b. $f(x)=1-2\sin x$

c. $f(x)=\dfrac{2}{1+7^x}$

d. $f(x)=\sqrt{-x^2-10x-9}$

B. Composite Function

5. Given $f(x) = \sqrt{x}$, $g(x) = x^2$, and $h(x) = x + 1$, write each function.

a. $g(h(f(x)))$ b. $f(h(g(x)))$

6. Write each function as a composite of elementary functions.

a. $f(x) = 5 - \dfrac{1}{\sqrt{x+1}}$

b. $f(x) = \log_3(\dfrac{1}{x^2+5})$

C. Inverse of a Function

7. Find the inverse of each function.

a. $f: \mathbb{R} \to \mathbb{R},\ f(x) = \dfrac{2-x}{5}$

b. $f: \mathbb{R} - \{3\} \to \mathbb{R} - \{2\},\ f(x) = \dfrac{-2x-3}{x+3}$

c. $f: \mathbb{R} \to \mathbb{R}^+,\ f(x) = 2 \cdot 3^{3x+4}$

d. $f: \mathbb{R}^+ \to \mathbb{R},\ f(x) = 2 + \ln(5x - 4)$

e. $f: [1, +\infty) \to [1, +\infty),\ f(x) = 4x^2 - 8x + 5$

8. For each function, find the real number a which satisfies the equation $f(a) = f^{-1}(a)$.

a. $f(x) = 5x - 2$

b. $f(x) = x^3 - 6x^2 + 13x - 8$

D. Constant, Increasing and Decreasing Functions

9. $f(x) = \dfrac{ax^2 + 2x + b}{3x^2 + bx + 2b}$ is a constant function. Find a and b.

10. Determine whether each function is increasing or decreasing on the given interval.

a. $f(x) = x^2 - 6x + 1$, $x \in (3, \infty)$

b. $f(x) = -x^2 + 4x - 3$, $x \in (-\infty, 2)$

c. $f(x) = -x^3 + 3$, $x \in \mathbb{R}$

d. $f(x) = \sin x$, $x \in (0, \dfrac{\pi}{2})$

11. The function $f(x) = 2x^2 - 7x - 15$ is given.

a. On which interval does the function decrease?

b. On which interval does the function increase?

E. Even and Odd Functions

12. Determine whether each function is even, odd, or neither even nor odd.

a. $f(x) = x^5 + x^3 + x$

b. $f(x) = \dfrac{\cos x + x^2}{3 + x^4}$

c. $f(x) = (\dfrac{x \cdot \tan x}{x^3 + \sin x})^5$

d. $f(x) = \sin(\tan(x^3 + x))$

e. $f(x) = e^{\frac{5x^3+x}{\cos x}}$

f. $f(x) = x^4 \cdot \sin x$

2 SPECIAL FUNCTIONS

A. PIECEWISE FUNCTION

Definition

piecewise function

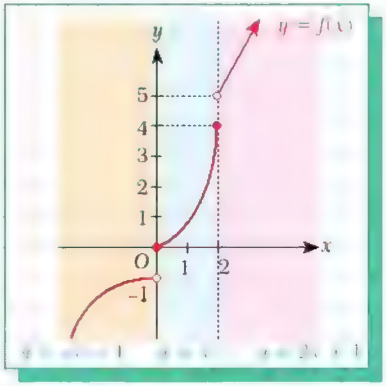
A function that is defined by different formulas on different intervals of its domain is called a piecewise function.

EXAMPLE 28 The piecewise function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 2x+1 & \text{if } x > 2 \\ x^2 & \text{if } 0 \le x \le 2 \\ -x^2-1 & \text{if } x < 0 \end{cases}$ is given.

a. Draw the graph of f.

b. Find $f(-5) + f(2) + f(3)$.

Solution a. When $x > 2$, we draw the graph of $y = 2x + 1$, when $0 \le x \le 2$, we draw the graph of $y = x^2$ and when $x < 0$, we draw the graph of $y = -x^2 - 1$.



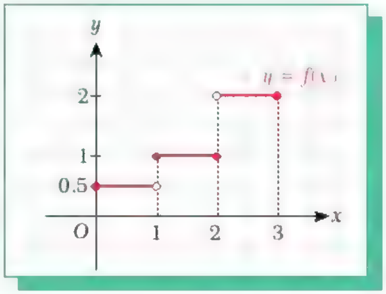
b. When $x = -5$, $f(x) = -x^2 - 1$. So $f(-5) = -(-5)^2 - 1 = -26$.

When $x = 2$, $f(x) = x^2$. So $f(2) = 2^2 = 4$.

When $x = 3$, $f(x) = 2x + 1$. So $f(3) = 2 \cdot 3 + 1 = 7$.

Hence, $f(-5) + f(2) + f(3) = -26 + 4 + 7 = -15$.

EXAMPLE 29 The domain of the function $f(x)$ shown in the figure is $[0, 3]$. Define $f(x)$ as a piecewise function.



Solution The graph consists of three line segments. Working from left to right:

the first line segment is valid for $0 \le x < 1$ and $f(x) = \dfrac{1}{2}$,

the second segment is valid for $1 \le x \le 2$ and $f(x) = 1$,

and the third segment is valid for $2 < x \le 3$ and $f(x) = 2$.

So the definition is

$$f(x) = \begin{cases} \dfrac{1}{2} & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \le 2 \\ 2 & \text{if } 2 < x \le 3. \end{cases}$$

EXAMPLE 30 Sketch the graph of the piecewise function $f: \mathbb{R} \to \mathbb{R}$,

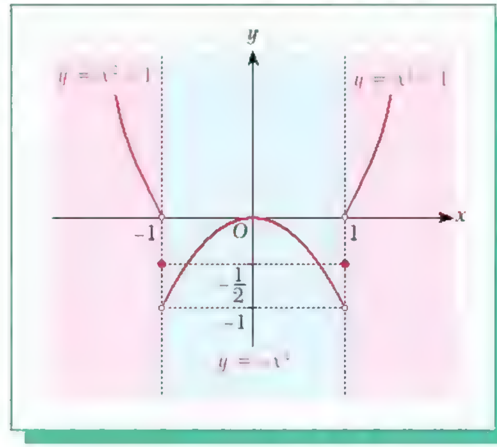
$$f(x)=\begin{cases} x^2-1 & \text{if } x<-1 \text{ or } x>1 \\ -\dfrac{1}{2} & \text{if } x=-1 \text{ or } x=1 \\ -x^2 & \text{if } -1<x<1. \end{cases}$$

Solution We draw the graph of $y = x^2 - 1$ for the interval $(-\infty, -1) \cup (1, \infty)$.

Since $f(-1) = -\dfrac{1}{2}$ and $f(1) = -\dfrac{1}{2}$,

we plot the single points $(-1, -\dfrac{1}{2})$ and $(1, -\dfrac{1}{2})$.

We draw the curve $y = -x^2$ for the interval $(-1, 1)$.



Check Yourself 6

1. The piecewise function $f: \mathbb{R} \to \mathbb{R}$, $f(x)=\begin{cases} x^2 & \text{if } x>3 \\ 3x+4 & \text{if } 0 \le x \le 3 \\ x^3+2 & \text{if } x<0 \end{cases}$ is given.

   Calculate $f(f(f(-1)))$.

2. Sketch the graph of each piecewise function.

   a. $f: \mathbb{R} \to \mathbb{R}$, $f(x)=\begin{cases} x & \text{if } x \ge 1 \\ -x & \text{if } x<1 \end{cases}$

   b. $f: \mathbb{R} \to \mathbb{R}$, $f(x)=\begin{cases} x^2 & \text{if } x>1 \\ -x^2 & \text{if } x \le 1 \end{cases}$

   c. $f: \mathbb{R} \to \mathbb{R}$, $f(x)=\begin{cases} 2x-1 & \text{if } x<0 \\ 1 & \text{if } 0 \le x<1 \\ 1-x & \text{if } x \ge 1 \end{cases}$

   d. $f: \mathbb{R} \to \mathbb{R}$, $f(x)=\begin{cases} \ln x & \text{if } 0<x \le e \\ e^x & \text{if } x \le 0 \text{ or } x>e \end{cases}$

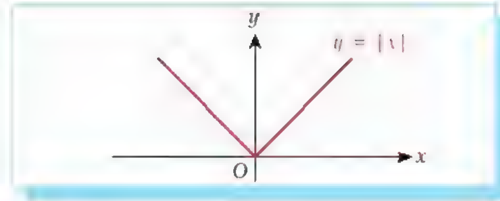
Answers

1. 49

B. ABSOLUTE VALUE FUNCTION

Recall that for any number x, the absolute value of x (written $|x|$) is the distance between x and the origin on a number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Definition

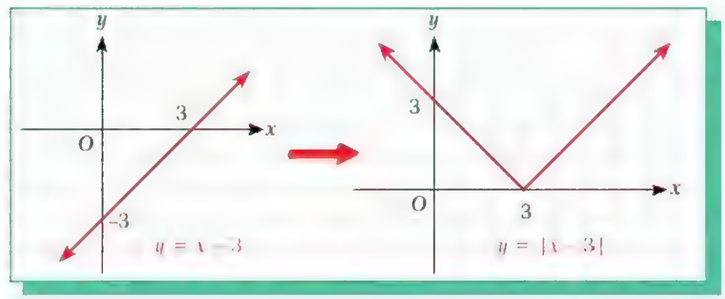
absolute value function

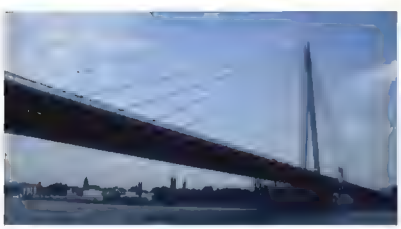
The absolute value function $|f(x)|$ is defined as

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0. \end{cases}$$

EXAMPLE 31 Draw the graph $y = |x - 3|$.

Solution We begin by drawing the graph $y = x - 3$. We then draw the graph $y = |x - 3|$ by reflecting the negative part of the graph.





Note

When solving absolute value equations or inequalities or when drawing the graph of an absolute value function, begin by finding the intervals in which the value of the function is negative, positive or zero.

EXAMPLE 32 Draw the graph $y = |x^2 - 1|$.

Solution 1 Let us construct the sign table for $x^2 - 1$.

x		-1		1	
$x^2 - 1$	+	0	–	0	+

$x^2 - 1$ is positive for $x < -1$ or $x > 1$ and zero for $x = -1$ or $x = 1$.

For these values of x, $|x^2 - 1| = x^2 - 1$.

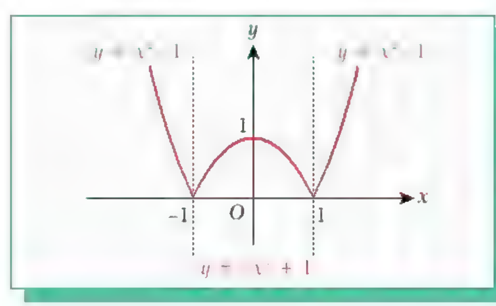
$x^2 - 1$ is negative for $-1 < x < 1$.

For these values of x,

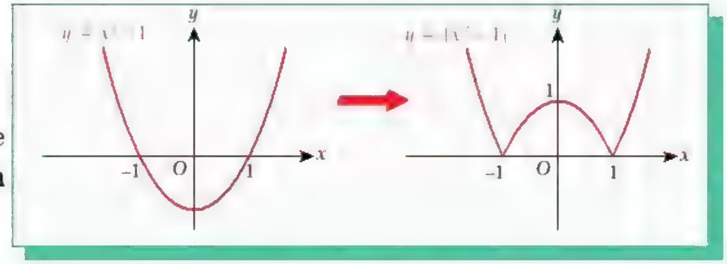
$|x^2 - 1| = -|x^2 - 1| = -x^2 + 1$.

$$\text{So } |x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } x \leq -1 \text{ or } x \geq 1 \\ -x^2 + 1 & \text{if } -1 < x < 1. \end{cases}$$

Now we can draw the graph, shown opposite.



Solution 2 We could also graph $y = |x^2 - 1|$ using a different method: first we graph $y = x^2 - 1$ and then we reflect the negative y-values in the graph with respect to the x-axis.



EXAMPLE 33 Draw the graph of $f(x) = |x - 1| + |x + 2|$.

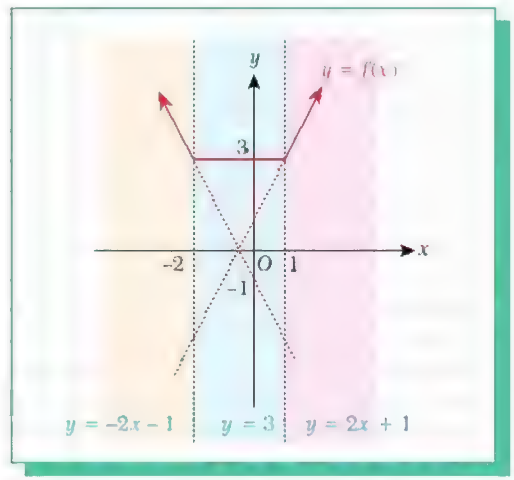
Solution First we construct a sign table.

x		-2		1	
$x - 1$	$-$		$-$	0	$+$
$x + 2$	$-$	0	$+$		$+$

When $x < -2$, $f(x) = -x + 1 - x - 2 = -2x - 1$.
When $-2 \le x < 1$, $f(x) = -x + 1 + x + 2 = 3$.
When $x \ge 1$, $f(x) = x - 1 + x + 2 = 2x + 1$.
So $f(x)$ can be defined piecewise as

$$f(x) = \begin{cases} -2x - 1 & \text{if} \quad x < -2 \\ 3 & \text{if} \quad -2 \le x < 1 \\ 2x + 1 & \text{if} \quad x \ge 1. \end{cases}$$

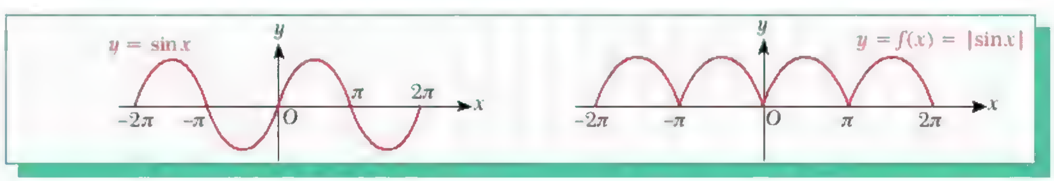
Now we can draw the graph, shown opposite.



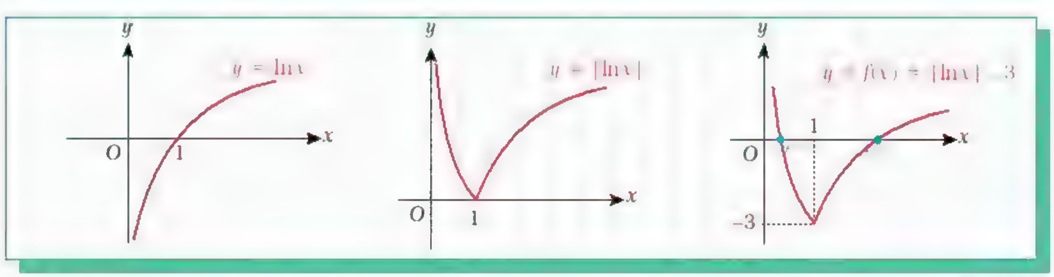
EXAMPLE 34 Draw the graph of each absolute value function.

a. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |\sin x|$ b. $f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = |\ln x| - 3$

Solution a.



b.



EXAMPLE 35 Draw the graph of $f(x) = -x \cdot |x+2| + 3x$.

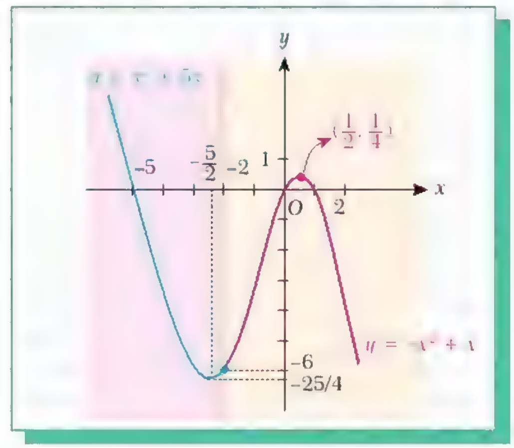
Solution Since the expression includes an absolute value, let us begin by defining the function in pieces:

x		-2	
$x+2$	$-$	0	$+$
$\lvert x+2 \rvert$	$-x-2$	0	$x+2$

$$f(x) = \begin{cases} -x(-x-2)+3x & \text{if} \quad x<-2 \\ -x(x+2)+3x & \text{if} \quad x \geq -2 \end{cases}$$

$$f(x) = \begin{cases} x^2+5x & \text{if} \quad x<-2 \\ -x^2+x & \text{if} \quad x \geq -2. \end{cases}$$

To draw the graph precisely, let us find the x-intercepts by calculating the roots of the equations:



$x^2 + 5x = 0$ and $-x^2 + x = 0$

$x(x+5) = 0$ $\quad$ $-x(x-1) = 0$

$x_1 = 0,\ x_2 = -5$ $\quad$ $x_3 = 0,\ x_4 = 1.$

At $x = -2$, $f(-2) = -(-2)^2 + (-2) = -6$ and also $(-2)^2 + 5 \cdot (-2) = -6$.

The vertex point of $y = x^2 + 5x$ is $(-\frac{5}{2}, -\frac{25}{4})$

and the vertex point of $y = -x^2 + x$ is $(\frac{1}{2}, \frac{1}{4})$.

Now we can draw the graph.

Check Yourself 7

Draw the graphs.

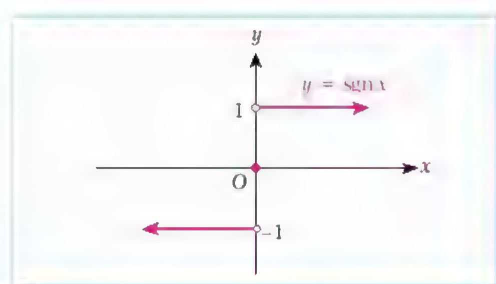
1. $y = |2x-3|$
2. $y = |x^3|$
3. $y = |x^2 + 2x - 3|$
4. $y = |x-5| + |x+3|$
5. $y = |x-5| \cdot x + 2x - 1$
6. $y = |2x+1| + x - 3$

C. SIGN FUNCTION

Definition **sign function**

Let $f: \mathbb{R} \to \mathbb{R}$ be a function.

The sign function $\operatorname{sgn} f(x)$ is defined as



$$\operatorname{sgn} f(x) = \begin{cases} 1 & \text{if} \quad f(x) > 0 \\ 0 & \text{if} \quad f(x) = 0 \\ -1 & \text{if} \quad f(x) < 0. \end{cases}$$

The sign function helps us to determine the sign of a function on different intervals.

The sign function is also called the signum function.

For example, if $x = -5$ then $\operatorname{sgn} x = -1$, if $x = 0$, $\operatorname{sgn} x = 0$ and if $x = 7$, $\operatorname{sgn} x = 1$.

EXAMPLE 36 Write the function $f(x) = \text{sgn}(3 - x)$ as a piecewise function.

Solution Let us construct the sign table of $(3 - x)$:

x		3	
$3 - x$	+	0	–
$\text{sgn}(3 - x)$	1	0	–1

Hence, $\text{sgn}(3 - x) = \begin{cases} 1 & \text{if } x < 3 \\ 0 & \text{if } x = 3 \\ -1 & \text{if } x > 3. \end{cases}$

EXAMPLE 37 Solve the equations.

a. $\text{sgn}(2x + 1) = -1$ b. $\text{sgn}(x^2 + 5x) = 0$ c. $\text{sgn}(\frac{2x}{x+1}) = 1$

Solution a. If $\text{sgn}(2x + 1) = -1$ then $2x + 1 < 0$, so $x < -\frac{1}{2}$.

b. If $\text{sgn}(x^2 + 5x) = 0$ then $x^2 + 5x = 0$, so $x = -5$ or $x = 0$.

c. If $\text{sgn}(\frac{2x}{x+1}) = 1$ then $\frac{2x}{x+1} > 0$, so $x \in \mathbb{R} - [-1, 0]$.

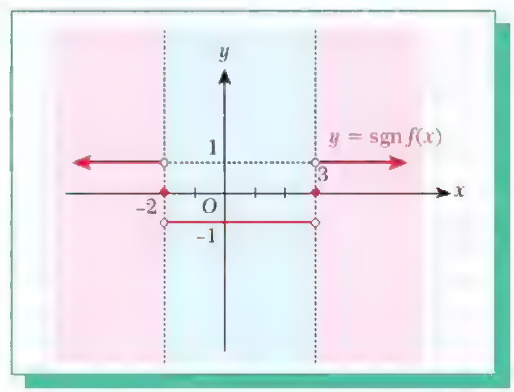
EXAMPLE 38 $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - x - 6$ is given. Draw the graph of $\text{sgn}(f(x))$.

Solution Let us construct the sign table of $x^2 - x - 6$:

x		–2		3	
$x^2 - x - 6$	+	0	–	0	+
$\text{sgn}(x^2 - x - 6)$	1	0	–1	0	1

So $\text{sgn} f(x) = \begin{cases} 1 & \text{if } x < -2 \text{ or } x > 3 \\ 0 & \text{if } x = -2 \text{ or } x = 3 \\ -1 & \text{if } -2 < x < 3. \end{cases}$

The graph of $y = \text{sgn} f(x)$ is shown opposite.



Check Yourself 8

1. Solve the equations.

   a. $\text{sgn}(x^2 - 1) = 0$ b. $\text{sgn}(2 - x) = 1$ c. $\text{sgn}(\ln x) = -1$ d. $\text{sgn}(\frac{x}{x-1}) = 1$

2. Draw the graph of each function.

   a. $f(x) = \text{sgn}(2 - x)$ b. $f(x) = \text{sgn}(\cos x)$, $x \in [-2\pi, 2\pi]$
   c. $f(x) = \text{sgn}(x^2 - 5x + 6)$ d. $f(x) = x - \text{sgn}(x)$

Answers

1. a. $\{-1, 1\}$ b. $x < 2$ c. $(0, 1)$ d. $\mathbb{R} - [0, 1]$

D. FLOOR FUNCTION

For any real number x, the greatest integer that is less than or equal to x is called the floor of x, denoted by $[\![x]\!]$.

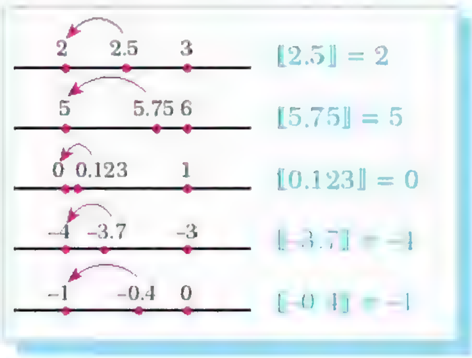
Definition

floor function

Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Then the floor function $[\![f(x)]\!]$ is defined as

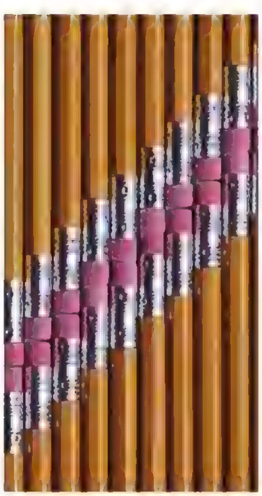
$$[\![f(x)]\!] = \begin{cases} f(x) & \text{if } \; f(x) \in \mathbb{Z} \\ \text{the greatest integer which is less than } f(x) & \text{if } \; f(x) \notin \mathbb{Z}. \end{cases}$$

Look at some examples of the floor function:



Similarly, $[\![6]\!] = 6$, $[\![-5]\!] = -5$, $[\![0.85]\!] = 0$, $[\![-2.5]\!] = -3$, $[\![\pi]\!] = 3$, $[\![e]\!] = 2$, $[\![-0.6]\!] = -1$ and $[\![-5.003]\!] = -6$.

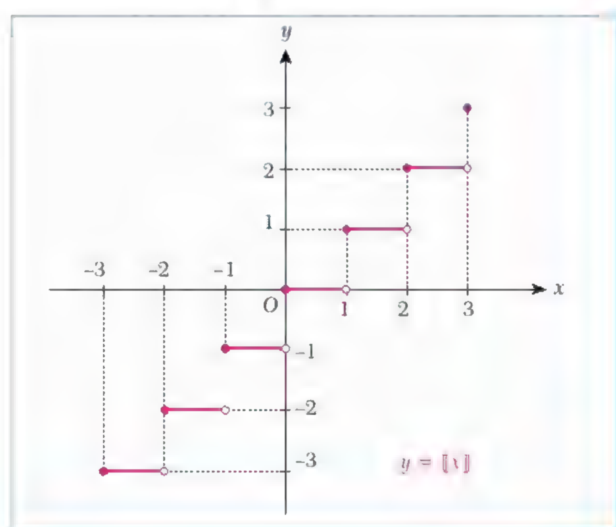
The floor function $f: \mathbb{R} \to \mathbb{Z}$, $f(x) = [\![x]\!]$ is also called the greatest integer function.



Now let us draw the graph $y = [\![x]\!]$ in the domain $[-3, 3]$.

Since the value $y = [\![x]\!]$ changes for each integer value of x, the integer values of x are important points for the function $y = [\![x]\!]$. We say that these values are the crucial points of $[\![x]\!]$. To draw the graph we have to consider the value of y between each of these crucial points:

if $-3 \le x < -2$,	$y = [\![x]\!] = -3$;
if $-2 \le x < -1$,	$y = [\![x]\!] = -2$;
if $-1 \le x < 0$,	$y = [\![x]\!] = -1$;
if $0 \le x < 1$,	$y = [\![x]\!] = 0$;
if $1 \le x < 2$,	$y = [\![x]\!] = 1$;
if $2 \le x < 3$,	$y = [\![x]\!] = 2$;
if $x = 3$,	$y = [\![x]\!] = 3$.



Remark

1. If $[\![x]\!] = t$ then we can write $t \le x < t + 1$ $(t \in \mathbb{Z})$.
2. For $a \in \mathbb{Z}$ and $x \in \mathbb{R}$, $[\![x + a]\!] = [\![x]\!] + a$.

For example, if $[\![x]\!] = 3$ then $3 \le x < 4$;
if $[\![x]\!] = -2$ then $-2 \le x < -1$;
if $[\![x]\!] = a$ then $a \le x < a + 1$;
if $[\![x + 5]\!] = 4$ then $[\![x]\!] + 5 = 4$, $[\![x]\!] = -1$ and so $-1 \le x < 0$.

EXAMPLE 39 Draw the graph of f: $[-4, 4] \to \mathbb{R}$, $f(x) = x \cdot [\![\frac{x}{2}]\!]$.

Solution First of all we have to find the crucial points at which $\frac{x}{2}$ takes an integer value.

For any real number x, if $[\![\frac{x}{2}]\!] = t$ then we can write $t \le \frac{x}{2} < t+1$, i.e. $2t \le x < 2t + 2$.

So the points $2t$ $(t \in \mathbb{Z})$ are the crucial points and we need to evaluate the function on the corresponding intervals:

if $-4 \le x < -2$ then $-2 \le \frac{x}{2} < -1$ and so $f(x) = x[\![\frac{x}{2}]\!] = x \cdot (-2) = -2x$;

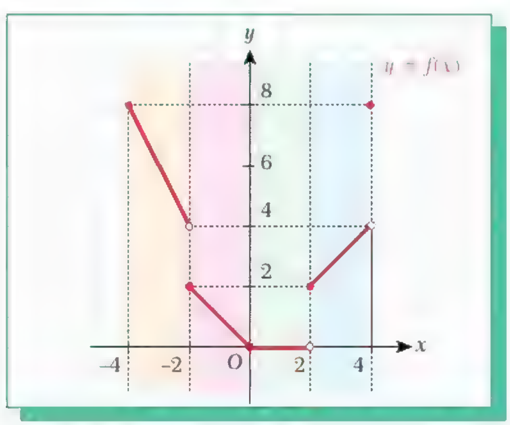
if $-2 \le x < 0$ then $-1 \le \frac{x}{2} < 0$ and so $f(x) = x[\![\frac{x}{2}]\!] = x \cdot (-1) = -x$;

if $0 \le x < 2$ then $0 \le \frac{x}{2} < 1$ and so $f(x) = x[\![\frac{x}{2}]\!] = x \cdot 0 = 0$;

if $2 \le x < 4$ then $1 \le \frac{x}{2} < 2$ and so $f(x) = x[\![\frac{x}{2}]\!] = x \cdot 1 = x$;

if $x = 4$ then $\frac{x}{2} = 2$ and so $f(x) = x[\![\frac{x}{2}]\!] = 4 \cdot 2 = 8$.

We can now draw the graph on each interval.



EXAMPLE 40 Draw the graph of $f: [-1, 1] \to \mathbb{R}$, $f(x) = [\![2x]\!] - x$.

Solution Let us find the crucial points of $[\![2x]\!]$.

If $[\![2x]\!] = t$, $t \le 2x < (t + 1)$ and so $\frac{t}{2} \le x < \frac{t}{2} + \frac{1}{2}$.

So the points $\frac{t}{2}$ $(t \in \mathbb{Z})$ are the crucial points and we evaluate the function on the corresponding intervals:

if $-1 \le x < -\frac{1}{2}$ then $-2 \le 2x < -1$ and so $f(x) = [\![2x]\!] - x = -2 - x$;

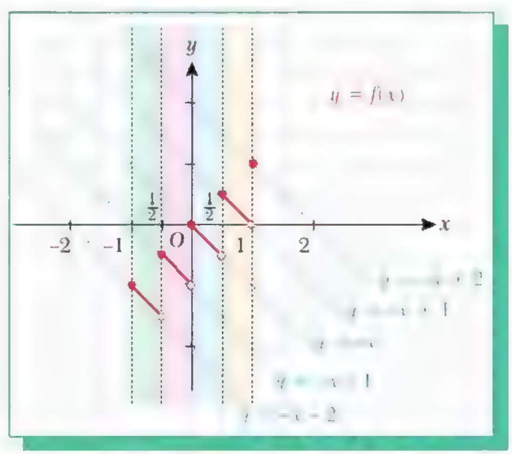
if $-\frac{1}{2} \le x < 0$ then $-1 \le 2x < 0$ and so $f(x) = [\![2x]\!] - x = -1 - x$;

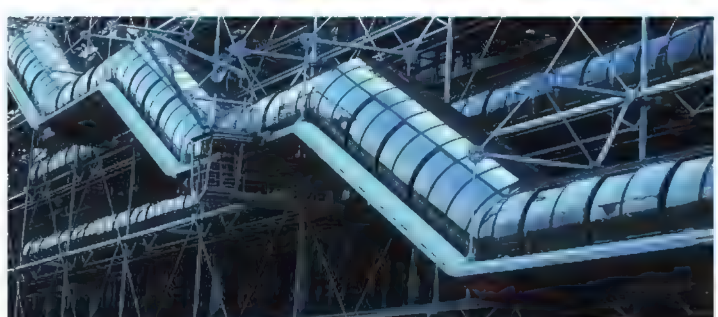
if $0 \le x < \frac{1}{2}$ then $0 \le 2x < 1$ and so $f(x) = [\![2x]\!] - x = 0 - x = -x$;

if $\frac{1}{2} \le x < 1$ then $1 \le 2x < 2$ and so $f(x) = [\![2x]\!] - x = 1 - x$;

if $x = 1$ then $2x = 2$ and so $f(x) = [\![2x]\!] - x = 2 - 1 = 1$.

We can now draw the graph on each interval.





EXAMPLE 41 Define the function f: $[-2, 2] \to \mathbb{R}$, $f(x) = [\![-x]\!] + x$ as a piecewise function and draw its graph.

Solution Let us find the crucial points of $[\![-x]\!]$.

If $[\![-x]\!] = t$ then $t \le -x < (t+1)$, i.e. $-t \ge x > (-t-1)$ and so $(-t-1) < x \le -t$.

So the points $t \in \mathbb{Z}$ are the crucial points.

We evaluate the function on the corresponding intervals:

if $x = -2$ then $-x = 2$ and so $f(x) = [\![-x]\!] + x = 2 + (-2) = 0$;

if $-2 < x \le -1$ then $2 > -x \ge 1$ and so $f(x) = [\![-x]\!] + x = 1 + x = x + 1$;

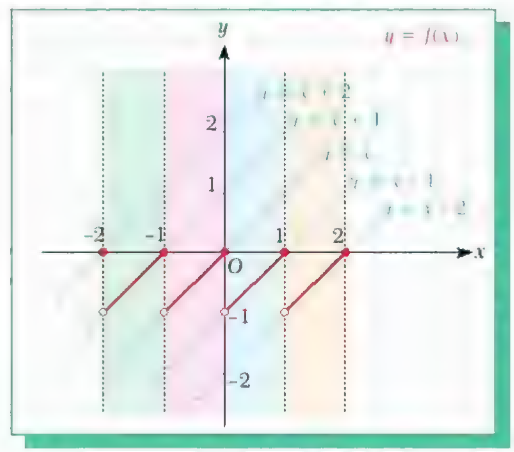
if $-1 < x \le 0$ then $1 > -x \ge 0$ and so $f(x) = [\![-x]\!] + x = 0 + x = x$;

if $0 < x \le 1$ then $0 > -x \ge -1$ and so $f(x) = [\![-x]\!] + x = -1 + x = x - 1$;

if $1 < x \le 2$ then $-1 > -x \ge -2$ and so $f(x) = [\![-x]\!] + x = -2 + x = x - 2$.

The piecewise function and the graph are as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = -2 \\ x+1 & \text{if } -2 < x \le -1 \\ x & \text{if } -1 < x \le 0 \\ x-1 & \text{if } 0 < x \le 1 \\ x-2 & \text{if } 1 < x \le 2. \end{cases}$$



Check Yourself 9

1. Find each value.

   a. $[\![-2.1]\!]$ b. $[\![-e]\!]$ c. $[\![0.9]\!]$ d. $[\![\sqrt{2}]\!]$ e. $[\![\pi]\!]$ f. $[\![-\pi]\!]$

2. Solve the equations.

   a. $[\![x-2]\!] = -3$ b. $[\![2x+1]\!] = 5$ c. $[\![\frac{3x-1}{4}]\!] = -1$ d. $[\![\ln x]\!] = 1$

3. Draw the graph of each function.

   a. $f(x) = [\![2x]\!]$ for $-1 \le x \le 1$ b. $f(x) = [\![\frac{x}{3}]\!]$ for $-5 \le x \le 4$ c. $f(x) = [\![-x]\!]$ for $-2 \le x \le 2$

Answers

2. a. $x \in [-1, 0)$ b. $x \in [2, \frac{5}{2})$ c. $x \in [-1, \frac{1}{3})$ d. $x \in [e, e^2)$

EXERCISES 1.2

A. Piecewise Function

1. Given

$$f(x)=\begin{cases} x-1 & \text{if} \quad x>2 \\ x & \text{if} \quad 0<x\le 2 \\ 4x & \text{if} \quad x\le 0 \end{cases}, \text{ find } \frac{f(0)+f(2)}{f(3)-f(1)}.$$

2. Draw the graph of each piecewise function.

a. $f(x)=\begin{cases} 1 & \text{if } x>1 \\ -2 & \text{if } x\le 1 \end{cases}$

b. $f(x)=\begin{cases} 2x+4 & \text{if } x>1 \\ -x & \text{if } x\le 1 \end{cases}$

c. $f(x)=\begin{cases} x^2+1 & \text{if } x>0 \\ x^2-1 & \text{if } x\le 0 \end{cases}$

d. $f(x)=\begin{cases} \cos x & \text{if } 0<x<\pi \\ \sin x & \text{if } \pi\le x<2\pi \end{cases}$

B. Absolute Value Function

3. Write each absolute value function as a piecewise function.

a. $f(x) = |x + 3|$

b. $f(x) = |x| + x$

c. $f(x) = |x^2 - x - 2|$

d. $f(x) = |x - 2| + |x - 3|$

e. $f(x) = |x + 1| + |x - 1|$

f. $f(x) = |x + 4| \cdot x + x^2 - 2x$

4. Draw the graphs.

a. $y = |-x|$ b. $y = |2 - 4x|$

c. $y = |x^2 - 1|$ d. $y = |x^2 - 4x - 5|$

e. $y = |\log x|$ f. $y = |\cos x|$

g. $y = x|x + 1| + 3$ h. $y = |5x + 4| + 2x - 1$

C. Sign Function

5. Solve the equations.

a. $\operatorname{sgn} x = 1$ b. $\operatorname{sgn}(x - 1) = -1$

c. $\operatorname{sgn}(x^2 - 1) = -1$ d. $\operatorname{sgn}(x^3 - x) = 0$

e. $\operatorname{sgn}(\log_{\frac{1}{2}} x) = -1$ f. $x + \operatorname{sgn} x = 3$

6. Draw the graphs.

a. $y = \operatorname{sgn} x^2$

b. $y = \operatorname{sgn}(x - 1)$

c. $y = \operatorname{sgn}\left(\frac{x+1}{x-2}\right)$

d. $y = \operatorname{sgn}(\sin x),\ x \in [-2\pi, 2\pi]$

e. $y = x + \operatorname{sgn} x$

f. $y = x^2 \operatorname{sgn} x$

D. Floor Function

7. Solve the equations.

a. $[\![x]\!] = -2$ b. $[\![x + 1]\!] = 3$

c. $[\![x - 1]\!] = -4$ d. $[\![\frac{2x+1}{3}]\!] = 1$

e. $[\![\frac{1-3x}{4}]\!] = 1$ f. $[\![\log_2 x]\!] = 3$

8. Draw the graph of each function on the given interval.

a. $f(x) = [\![x + 1]\!],\ x \in [-2, 2)$

b. $f(x) = [\![2 - x]\!],\ x \in [-3, 1]$

c. $f(x) = [\![2x - 1]\!],\ x \in [-1, 1]$

d. $f(x) = x[\![x]\!],\ x \in [-2, 2]$

e. $f(x) = x + [\![x]\!],\ x \in [-2, 2]$

Mixed Problems

9. Find the values of x for which each function is undefined.

a. $f(x)=\begin{cases}\dfrac{1}{x^2-16} & \text{if } x>0\\ \dfrac{1}{x^2-x-2} & \text{if } x\le 0\end{cases}$

b. $f(x)=\begin{cases}\dfrac{x^2}{x^2-9} & \text{if } x>0\\ \log_3(x^2-x) & \text{if } -2<x\le 0\\ \sqrt{x^2+3x-4} & \text{if } x\le -2\end{cases}$

10. f and g are two piecewise functions defined as

$f(x)=\begin{cases}x^2-1 & \text{if } x>2\\ 3x+1 & \text{if } x\le 2\end{cases}$ and

$g(x)=\begin{cases}x+1 & \text{if } x>0\\ x+2 & \text{if } x\le 0.\end{cases}$

Find an expression for the function $f(g(x))$.

11. Find the inverse of the piecewise function

$f:\mathbb{R}\to\mathbb{R},\ f(x)=\begin{cases}4x-1 & \text{if } x>1\\ 2x+1 & \text{if } x\le 1.\end{cases}$

12. Write each function as a piecewise function.

a. $f(x)=|1-x|$

b. $f(x)=|x^2-3x|-4$

c. $f(x)=|x^3|-x$

d. $f(x)=\left|\dfrac{3^x\cdot x^2}{x-4}\right|$

13. Write each function as a piecewise function.

a. $f(x)=x\cdot|x|$

b. $f(x)=x+|x-1|$

c. $f(x)=x^2-|x^2-2x-3|$

d. $f(x)=|x|+|x-1|$

e. $f(x)=|x-2|-|x-3|$

f. $f(x)=|x^2-1|+|x|$

14. Write each function as a piecewise function.

a. $f(x)=\operatorname{sgn}(x+2)-|x-1|$

b. $f(x)=|x|\cdot\operatorname{sgn}(1-x)$

c. $f(x)=|x-|x||-1$

d. $f(x)=\dfrac{|x|}{\operatorname{sgn}(x^2+1)}$

15. For $-1<x<2$, simplify the function

$f(x)=2x+\sqrt{x^2-4x+4}-\operatorname{sgn}(x+1)+[\![\dfrac{x+1}{3}]\!]$.

16. $f(x)=\begin{cases}x+\operatorname{sgn}(x-2) & \text{if } x\ge 0\\ [\![1-x]\!] & \text{if } x<0\end{cases}$, is given.

Find $(f\circ f)(-\dfrac{5}{2})$.

CHAPTER SUMMARY

- The domain of the polynomial function $f(x)=a_n x^n + a_{n-1}x^{n-1}+\ldots+a_0$ is $\mathbb{R}$.
- The domain of the rational function $f(x)=\dfrac{g(x)}{h(x)}$ is $\mathbb{R}-\{x \mid h(x)=0\}$.
- The domain of the radical function $f(x)=\sqrt[n]{g(x)}$ (n odd) is $\mathbb{R}$.
- The domain of the radical function $f(x)=\sqrt[n]{g(x)}$ (n even) is $\mathbb{R}-\{x \mid g(x)<0\}$.
- The domain of the exponential function $f(x)=a^x$ ($a\in\mathbb{R}^+-\{1\}$) is $\mathbb{R}$.
- The domain of the logarithmic function $f(x)=\log_a g(x)$ ($a\in\mathbb{R}^+-\{1\}$) is $\mathbb{R}-\{x \mid g(x)\le 0\}$
- A composite function such as $f(g(x))$ is a function formed by composing two or more elementary functions.
- If $f(x)=ax+b$ then $f^{-1}(x)=\dfrac{x-b}{a}$.
- If $f(x)=\dfrac{ax+b}{cx+d}$ then $f^{-1}(x)=\dfrac{-dx+b}{cx-a}$.
- The inverse of the exponential function $f(x)=a^x$ ($a\in\mathbb{R}^+-\{1\}$) is $f^{-1}(x)=\log_a x$.
- The inverse of the logarithmic function $f(x)=\log_a x$ ($a\in\mathbb{R}^+-\{1\}$) is $f^{-1}(x)=a^x$.
- Let $f: D\to R$ be a function. For all $x_1, x_2\in D$ such that $x_1<x_2$,
  a. if $f(x_1)=f(x_2)$ then f is called a constant function.
  b. if $f(x_1)<f(x_2)$ then f is called an increasing function.
  c. if $f(x_1)\le f(x_2)$ then f is called non-decreasing function.
  d. if $f(x_1)>f(x_2)$ then f is called a decreasing function.
  e. if $f(x_1)\ge f(x_2)$ then f is called non-increasing function.

  For any $x\in D$,
  a. if $f(-x)=f(x)$ then f is called an even function.
  b. if $f(-x)=-f(x)$ then f is called an odd function.

  The graph of an even function is symmetric with respect to the y-axis. The graph of an odd function is symmetric with respect to the origin.
- There are four special types of function:
  1. piecewise functions
  2. the absolute value function $|f(x)|$, defined as

$$f: \mathbb{R}\to\mathbb{R},\ |f(x)|=\begin{cases} f(x) & \text{if } f(x)\ge 0\\ -f(x) & \text{if } f(x)<0\end{cases}$$

  3. the sign function $\operatorname{sgn} f(x)$, defined as

$$f: \mathbb{R}\to\mathbb{R},\ \operatorname{sgn} f(x)=\begin{cases} 1 & \text{if } f(x)>0\\ 0 & \text{if } f(x)=0\\ -1 & \text{if } f(x)<0\end{cases}$$

  4. the floor function $[\![f(x)]\!]$, defined as $f: \mathbb{R}\to\mathbb{R}$,

$$[\![f(x)]\!]=\begin{cases} f(x) & \text{if } f(x)\in\mathbb{Z}\\ \text{the greatest integer less than } f(x) & \text{if } f(x)\notin\mathbb{Z}\end{cases}$$

  If $[\![x]\!]=t$ then $t\le x<(t+1)$, $(t\in\mathbb{Z})$.
  For $a\in\mathbb{Z}$ and $x\in\mathbb{R}$, $[\![x+a]\!]=[\![x]\!]+a$.

Concept Check

1. What is the definition of the domain of a function?
2. What is the difference between the domain and the range of a function?
3. What is the difference between the image and the range of a function?
4. What is the largest possible range of a function?
5. In which case is the domain of a radical function the set of real numbers?
6. What is the relation between the exponential and logarithmic functions?
7. What is the domain of a function which is the sum or difference of different types of function?
8. What is a composite function?
9. What is important about a one-to-one and onto function?
10. If a function is not one-to-one and onto, how can we find its inverse?
11. What is a non-decreasing function?
12. What is a non-increasing function?
13. What is the identity function? What is its inverse?
14. What is the difference between the graph of an even function and the graph of an odd function?
15. Can a function be even and odd at the same time?
16. Are there any functions which are neither even nor odd? Give an example.
17. Which one of the four basic trigonometric functions is an even function?
18. What are the four special types of function?
19. What do we call a function which is defined by different formulas over different parts of its domain?
20. How can we draw the graph of an absolute value function?
21. What is the definition of the sign function?
22. What is the definition of the floor function?
23. What are the crucial points of the floor of a function?

1. What is the domain of $f(x)=\dfrac{2x}{\log 2x}$?

A) $\mathbb{R}$ B) $(0, \infty)$ C) $(0, \infty)-\{\frac{1}{2}\}$

D) $(0, \infty)-\{2\}$ E) $(0, \frac{1}{2})$

2. What is the domain of $f(x)=\sqrt{|2x-1|-|3x+1|}$?

A) $[-2, 0]$ B) $(-2, 0)$ C) $[0, \infty)$

D) $(-\infty, -2]$ E) $\mathbb{R}$

3. What is the domain of $f(x)=\sqrt{\dfrac{2-x}{x+2}}$?

A) $[-4, -2)$ B) $[0, 2]-\{1\}$ C) $(-2, 2)$

D) $(-2, 2]$ E) $(-2, 0]$

4. What is the range of $f(x) = -x + 5$ for $x \in [-5, 5]$?

A) $[-5, 5]$ B) $[0, 5]$ C) $[-5, 10]$

D) $[5, 10]$ E) $[0, 10]$

5. What is the range of $f(x) = 2^{x+1}$ for $x \in (-1, 3)$?

A) $(2, 8)$ B) $(1, 16)$ C) $(7, 15)$

D) $(2, 16)$ E) $(0, 2)$

6. Find the inverse of $f: [1, \infty) \to [4, \infty)$, $f(x) = x^2 - 2x + 5$.

A) $\sqrt{x-4}$ B) $\sqrt{x-4}+1$ C) $\sqrt{x^2-2}+1$

D) $\sqrt{x-2}+1$ E) $\sqrt{x-4}-1$

7. Find the inverse of $f: \mathbb{R} \to (1, \infty)$, $f(x) = 3^{x-1} + 1$.

A) $\log_3 3x$ B) $\log_3(x-1)$ C) $\log_3(x-1)+2$

D) $\log_3(3x-3)$ E) $\log_3 x-1$

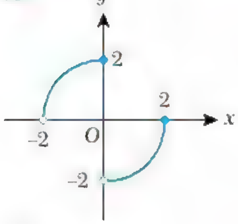
8. Which one of the following is an odd function?

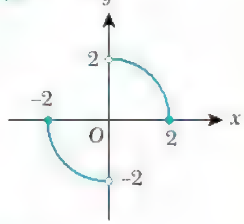
A) $f(x)=\dfrac{x^2}{|x|}$ B) $f(x)=\sqrt{5-x}+\sqrt{5+x}$

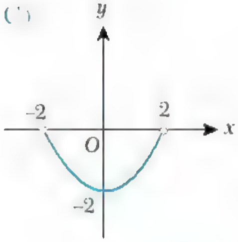
C) $f(x)=\dfrac{x^5}{x^3-x}$ D) $f(x) = x^3 + x$

E) $f(x)=\sqrt{\dfrac{x^2}{x-1}}$

9. Which one of the following is the graph of an odd function?

A)


B)


C)


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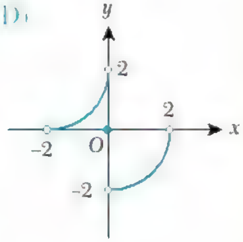
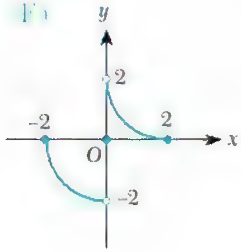


E)


10. Given that $f(x)=\dfrac{-2x+1}{ax+3}$ is a constant function, find a.

A) –2 B) –3 C) –4 D) –6 E) –10

11. Which one of the following functions is equivalent to $f(x)=x-\dfrac{x-1}{|x-1|}$ for $x<1$?

A) $g(x) = x + 1$ B) $g(x) = x - 1$

C) $g(x) = 2x + 1$ D) $g(x) = x - 2$

E) $g(x) = x + 2$

12. Which one of the following functions is equivalent to $f(x)=\begin{cases} x-5 & \text{if } x \geq 5 \\ -x+5 & \text{if } x<5 \end{cases}$?

A) $g(x) = |x| - 5$ B) $g(x) = |x| + 5$

C) $g(x) = |x - 5|$ D) $g(x) = |x + 5|$

E) $g(x) = |2x - 5|$

13. Solve $\operatorname{sgn}(\dfrac{x-1}{x+1})=-1$.

A) $x \in (-1, \infty)$ B) $x \in (-1, 1)$

C) $x \in (-\infty, 1)$ D) $x \in [-1, 1]$

E) $x \in (-1, 1]$

14. Solve $|x - 2| = \operatorname{sgn}(x - 2)$.

A) $x = 2$ B) $x = 3$ C) $x \in \{1, 2, 3\}$

D) $x \in \{2, 3\}$ E) $x \in \{0, 2\}$

15. Given $[\![5x - 1]\!] = 4$, find x.

A) $x\in[0, \dfrac{5}{4})$ B) $x\in[2, \dfrac{7}{6})$ C) $x \in [1, 6)$

D) $x\in[2, \dfrac{5}{4})$ E) $x\in[1, \dfrac{6}{5})$

16. How many different values does $f(x)=[\![\dfrac{x}{3}-1]\!]$ take in the interval $(-3, 7)$?

A) 5 B) 4 C) 3 D) 2 E) 1

1. What is the domain of $f(x)=\log(\frac{2x}{x+5})$?

A) $\mathbb{R}$ B) $\mathbb{R}-(-5, 2)$ C) $\mathbb{R}-[2, 5]$ D) $\mathbb{R}-[-5, 0]$ E) $\mathbb{R}-(0, 2)$

2. What is the domain of $f(x)=\sqrt[5]{x^2-4x-12}$?

A) $(-6, 2)$ B) $(-2, 6)$ C) $\mathbb{R}-(-2, 6)$ D) $\mathbb{R}^+$ E) $\mathbb{R}$

3. What is the domain of $f(x)=\frac{2x+1}{\sqrt{4-x^2}}$?

A) $[-2, 2]$ B) $[0, -2]$ C) $(-2, 2)$ D) $\mathbb{R}-(-2, 2)$ E) $\mathbb{R}$

4. What is the range of $f(x)=\sqrt{-x^2+5x+36}$ for the largest domain of f?

A) $[0, \frac{9}{2}]$ B) $[0, \frac{11}{2}]$ C) $[0, \frac{13}{2}]$ D) $(0, \frac{15}{2})$ E) $[0, \frac{17}{2})$

5. What is the range of $f(x)=\log_{\frac{1}{2}}(3x+1)$ for the largest domain of f?

A) $(-\frac{1}{3}, \infty)$ B) $(-1, 1)$ C) $(-\infty, \frac{1}{3})$ D) $\mathbb{R}$ E) $\mathbb{R}-\{-\frac{1}{3}\}$

6. Find the inverse of $f: \mathbb{R} \to \mathbb{R},\ f(x) = x^3-3x^2+3x$.

A) $\sqrt[3]{x}-1$ B) $\sqrt[3]{x-1}+1$ C) $\sqrt[3]{x+1}-1$ D) $\sqrt{x}+1$ E) $\sqrt{x-1}+1$

7. Find the inverse of

$f: \mathbb{R} \to (1, \infty),\ f(x) = 1+\log(3^x+1)$.

A) $\log_3(10^{x-1}-1)$ B) $\log(x-1)$ C) $\log_3(x-1)$ D) 10^{x-1} E) $\log 3^{x-1}$

8. Which one of the following is an even function?

A) $f(x) = x^7-x^3$ B) $f(x) = 2x$ C) $f(x) = x\sin x$ D) $f(x) = x\cos x$ E) $f(x) = \frac{\cos x}{x^3}$

9. Given that

$$f:\mathbb{R}-\{-1,\ 1\}\to\mathbb{R},\ f(x)=\frac{3x^2-(a-2)x+3}{1-x^2}$$

is an even function, find the value of $a^2 \cdot f(a)$.

A) 5 B) 10 C) –15 D) –20 E) 25

10. Given f: $[-a, a] \to \mathbb{R}$, which of the following must be an odd function?

A) $f(x)\cdot f(-x)$ B) $\dfrac{f(x)+f(-x)}{2}$

C) $\dfrac{f(x)-f(-x)}{2}$ D) $f(x)+f(\frac{1}{x})$

E) $f(x)-f(\frac{1}{x})$

11. Which one of the following functions is equivalent to $f(x)=\begin{cases}2x+2 & \text{if } x\geq -2\\ -2 & \text{if } x<-2\end{cases}$?

A) $g(x) = |x + 2|$ B) $g(x) = |x - 2|$

C) $g(x) = |x| + x + 2$ D) $g(x) = x - |x + 2|$

E) $g(x) = x + |x + 2|$

12. Solve $x^2 - 3x - 3 = \mathrm{sgn}(x^2 + 1)$.

A) $x = -1$ B) $x \in \{-1, 4\}$ C) $x \in \{-1, 1\}$

D) $x \in \{1, 4\}$ E) $x = 4$

13. Solve $\mathrm{sgn}(x + 1) = [\![x + 1]\!]$.

A) $x \in [-2, -1] \cup [0, 1)$ B) $x \in [-2, -1]$

C) $x \in [0, 1)$ D) $x \in [-2, 2]$

E) $x \in [-2, 0] \cup [1, 2]$

14. Solve $|5x - 3| \geq [\![\mathrm{sgn}\, x]\!]$.

A) $[\frac{2}{5}, \frac{4}{5}]$ B) $(-\infty, \frac{2}{5})$ C) $[\frac{4}{5}, \infty)$

D) $\mathbb{R}-(\frac{2}{5}, \frac{4}{5})$ E) $\mathbb{R}$

15. Solve $[\![\frac{2x-1}{5}]\!] = -2$.

A) $x \in [-2, 1)$ B) $x \in [1, 2)$

C) $x \in [-\frac{9}{2}, -2)$ D) $x \in [-\frac{7}{2}, -1)$

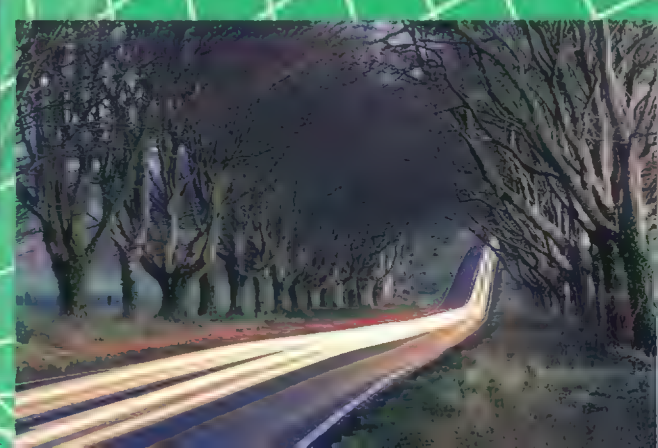
E) $x \in [-\frac{5}{2}, 3)$

16. How many different values can $f(x) = [\![5x + 3]\!]$ take in the interval $[2, 3)$?

A) 3 B) 4 C) 5 D) 6 E) 7

CHAPTER 2

LIMIT OF A FUNCTION



1 DEFINITION

A. LIMIT OF A POLYNOMIAL FUNCTION

Consider the polynomial function $f(x) = 2x$. We are asked to investigate what happens to the value of $f(x)$ as x gets closer to 2. We could begin by choosing a value of x which is close to 2, for example 1.5. We can calculate $f(1.5) = 2 \cdot 1.5 = 3$. Now we choose a value which is closer to 2, for example 1.75: $f(1.75) = 3.5$. Continuing like this, we can make a table of values of $f(x)$ as x gets closer to 2.

x	1.5	1.75	1.8	1.9	1.95	2	2.05	2.1	2.2	2.25	2.5
$2x$	3	3.5	3.6	3.8	3.9		4.1	4.2	4.4	4.5	5

Using this table, we can guess that as x gets closer to (i.e. *approaches*) 2, the value of $f(x)$ approaches 4. We say that 4 is the limit of $f(x) = 2x$ as x approaches 2, and write $\lim\limits_{x\to 2}(2x) = 4$. In this notation, the arrow symbol ($\to$) means 'approaches'. $x \to 2$ means x approaches the number 2.

Notice that for $f(x) = 2x$, $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} 2x = 4$, which is the same as $f(2)$. Similarly, we can calculate $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} 2x = 6 = f(3)$ and $\lim\limits_{x\to 12} f(x) = \lim\limits_{x\to 12} 2x = 24 = f(12)$, etc. In other words, in each case $\lim\limits_{x\to c} f(x) = f(c)$. In fact, this result is true for any polynomial function.

Definition

limit of a polynomial function

The limit of a polynomial function $f(x)$ as x approaches a point c is $f(c)$: $\lim\limits_{x\to c} f(x) = f(c)$.

In other words, for $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, $\lim\limits_{x\to c} f(x) = f(c) = a_n c^n + a_{n-1}c^{n-1} + \ldots + a_0$.

For example, let us calculate the limit of $f(x) = 2x$ when x approaches 5:

$\lim\limits_{x\to 5} f(x) = \lim\limits_{x\to 5} 2x = 2 \cdot 5 = 10$.

EXAMPLE 1 Calculate the limits.

a. $\lim\limits_{x\to 2}(4x-1)$ b. $\lim\limits_{x\to -1}(x^2+3x+2)$ c. $\lim\limits_{t\to 4}(3t-t^2)$

Solution These are all polynomial functions, so we can use $\lim\limits_{x\to c} f(x) = f(c)$.

a. $\lim\limits_{x\to 2}(4x-1) = 4\cdot 2 - 1 = 7$

b. $\lim\limits_{x\to -1}(x^2+3x+2) = (-1)^2 + 3\cdot(-1) + 2 = 1 - 3 + 2 = 0$

c. $\lim\limits_{t\to 4}(3t-t^2) = 3\cdot 4 - 4^2 = 12 - 16 = -4$.

Check Yourself 1

1. Given $f(x) = 4x - 1$, complete the table to find the limit of $f(x)$ as x approaches 3.

x	2.5	2.75	2.8	2.9	2.95	3	3.05	3.1	3.2	3.25	3.5
$f(x)$	9									12	

2. Calculate the limit of each polynomial function.

a. $\lim\limits_{x\to 0} 5x$ b. $\lim\limits_{x\to 3} x(2-x)$ c. $\lim\limits_{x\to 6}(x^2 - 3x - 15)$

d. $\lim\limits_{x\to a} 2x(x+1)$ e. $\lim\limits_{t\to -3} 3t^2(2t-1)$ f. $\lim\limits_{x\to 3} 7$

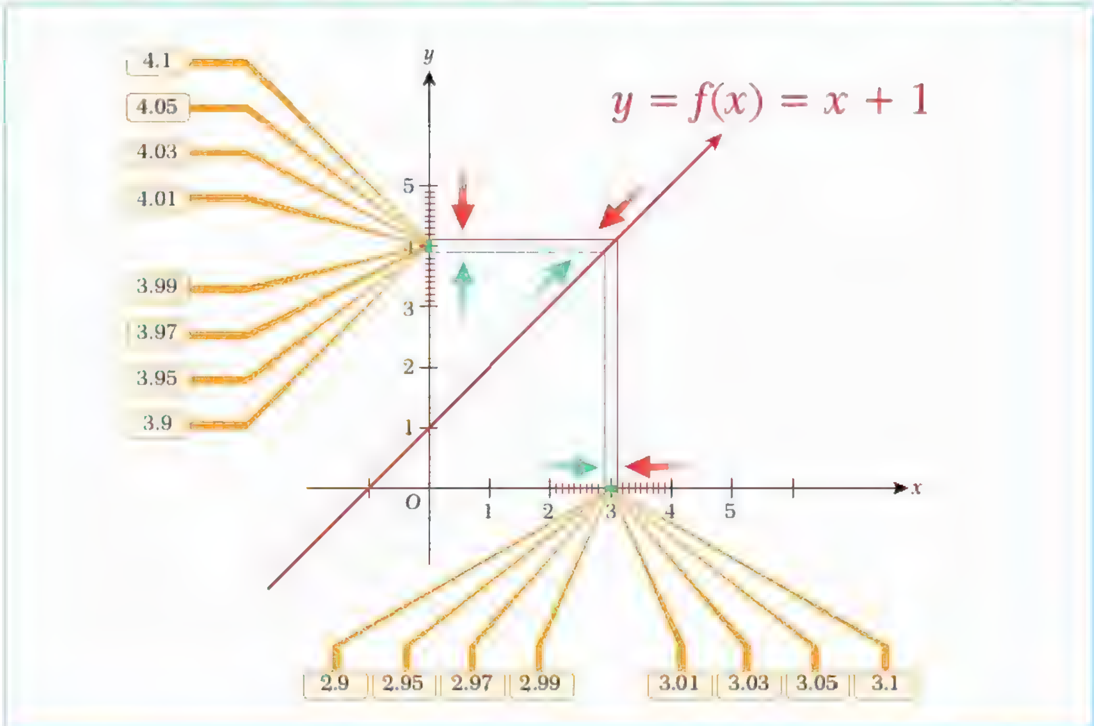
Answers

2. a. 0 b. –3 c. 3 d. $2a^2 + 2a$ e. –189 f. 7

B. LIMITS ON A GRAPH

We can also use the graph of a function to study its limit as x approaches a certain point. For example, let us draw the graph of the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$ and use it to find $\lim\limits_{x\to 3} f(x)$.

The notation $f: \mathbb{R} \to \mathbb{R}$ means a function from $\mathbb{R}$ (the domain) to $\mathbb{R}$ (the range).
The notation $\lim\limits_{x\to 3} f(x)$ means the limit of $f(x)$ as x approaches 3.

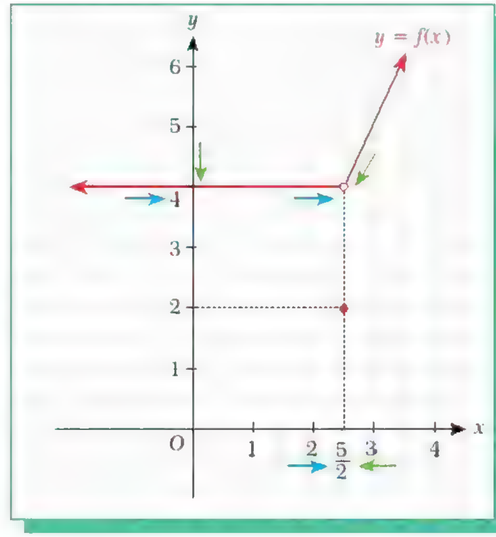


We can approach the point $x = 3$ from two directions: the right (as the values of x get gradually smaller) and the left (as the values of x get gradually bigger).

In both cases, the limit of $f(x) = x + 1$ as x approaches 3 is 4: $\lim\limits_{x\to 3}(x+1) = 3+1 = 4$.

EXAMPLE 2 Given the piecewise function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 4 & \text{if } x < \frac{5}{2} \\ 2 & \text{if } x = \frac{5}{2} \\ 2x-1 & \text{if } x > \frac{5}{2} \end{cases}$, find $\lim\limits_{x \to \frac{5}{2}} f(x)$.

Solution Let us first draw the graph of the function. As we can see, $x = \frac{5}{2}$ is a crucial point in the graph. When we approach $\frac{5}{2}$ from the right-hand side (i.e. when x is greater than $\frac{5}{2}$) we use the function $f(x) = 2x - 1$ and get $\lim\limits_{x \to \frac{5}{2}} (2x-1) = 4$.



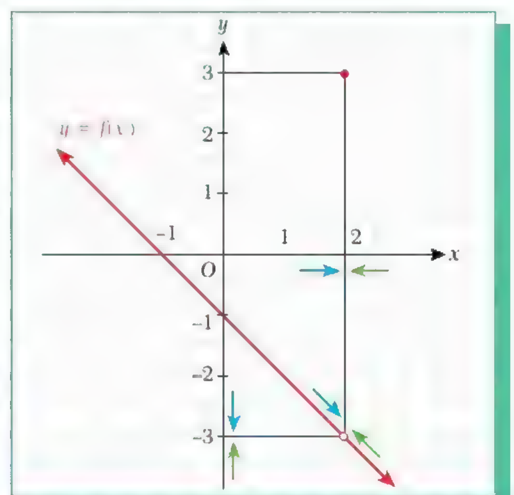
A point at which we need to check the right-hand and the left-hand limits of a function is called a crucial point of the function.

So we can say that f approaches 4 from the right-hand side. When we approach x from the left-hand side (i.e. when x is less than $\frac{5}{2}$) we use the function $f(x) = 4$, which is constant. Its limit is 4 when x approaches from the left-hand side. As a result, f approaches 4 as x approaches $\frac{5}{2}$ from both sides, i.e. $\lim\limits_{x \to \frac{5}{2}} f(x) = 4$.

EXAMPLE 3 Given the piecewise function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} -x-1 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$, find $\lim\limits_{x \to 2} f(x)$.

Solution Let us draw the graph of $f(x)$.

$x = 2$ is a crucial point. Notice that $f(2) = 3$ but 3 is not the limit of f at 2. This is because the limit is the value which $f(x)$ approaches as x approaches 2. And in the graph we can see that the limit of $f(x)$ when x approaches 2 is –3.

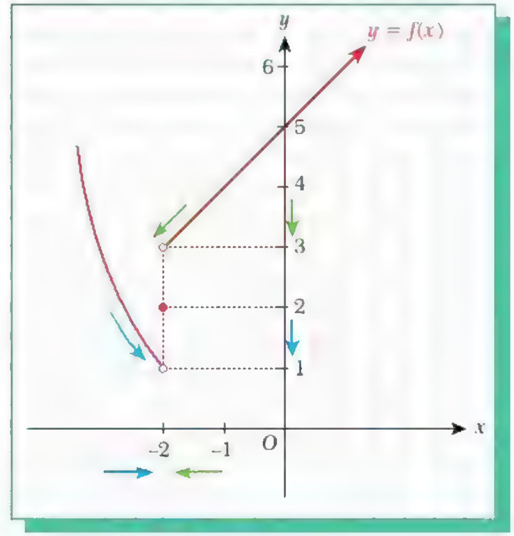


As x gets closer to a point c, although the limit exists and approaches a number, at the point c a function may have a different value, or may not even be defined. What happens at the given point is not important for the limit at this point.

EXAMPLE 4 A piecewise function $f(x)$ is given as $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^2 - 3 & \text{if} \quad x < -2 \\ 2 & \text{if} \quad x = -2 \\ x + 5 & \text{if} \quad x > -2 \end{cases}$.

Find $\lim_{x \to -2} f(x)$.

Solution First we draw a graph of the function. Notice that $x = -2$ is a crucial point. We need to examine what happens at the point $x = -2$ when we approach it from the left and from the right.



When we approach –2 from the right-hand side, the function is $f(x) = x + 5$ and $\lim_{x \to -2} x + 5 = 3$, so f approaches 3.

On the other hand, when we approach –2 from the left-hand side, x is less than –2 and we use $f(x) = (x^2 - 3)$ so $\lim_{x \to -2} (x^2 - 3) = 1$.

We can see that we get different results if we approach $x = -2$ from different sides. For this reason, we say that at this point the limit does not exist.

Note

If we get different results for a limit when we approach it from the right and from the left, we say that the limit does not exist at this point.

Check Yourself 2

Graph each function and evaluate the given limit.

1. $f: \mathbb{R} \to \mathbb{Z}$, $f(x) = 3$, $\lim_{x \to 2} f(x)$
2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = -x - 2$, $\lim_{x \to 3} f(x)$
3. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^2 & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$, $\lim_{x \to -2} f(x)$
4. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^2 - 1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \\ x^2 + 1 & \text{if } x < 1 \end{cases}$, $\lim_{x \to 1} f(x)$

Answers

1. 3 2. –5 3. 4 4. does not exist

C. DEFINITION OF LIMIT

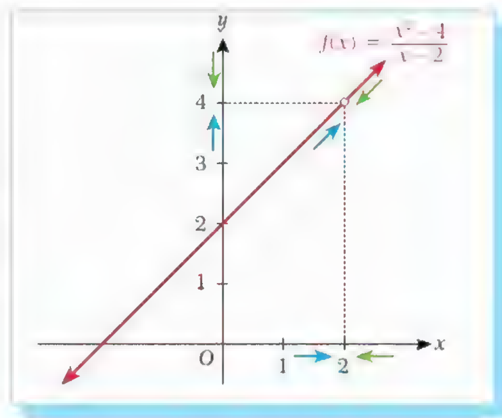
We have seen how to calculate the limit of a polynomial function, and we have used graphs to calculate the limits of some other functions. However, we still do not have a general formula for the limit of a function.

For example, consider the limit of the function $f\colon \mathbb{R} - \{2\} \to \mathbb{R}$, $f(x) = \dfrac{x^2-4}{x-2}$ when x approaches 2, i.e. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$. $f(x)$ is not a polynomial function and the graph of $f(x)$ will be difficult to draw. Also, $f(x) = \dfrac{x^2-4}{x-2}$ is undefined at $x = 2$.

However, remember that when we calculate a limit we are examining the value of a function as it *approaches* a point, not the value at the point itself. So by applying simple factorization methods we can get

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)}$$

$$= \lim_{x\to 2}(x+2) = 2+2 = 4.$$



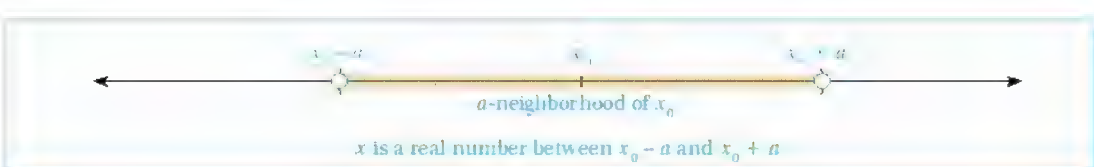
We can say that $\dfrac{x-2}{x-2} = 1$ because x is very close to 2 but not equal to 2, so $\dfrac{x-2}{x-2}$ is not zero divided by zero and so it can be simplified. As we can see in the graph of the function

$f\colon \mathbb{R} - \{2\} \to \mathbb{R}$, $f(x) = \dfrac{x^2-4}{x-2}$, the limit of $f(x)$ as x approaches 2 is 4.

1. Neighborhood of a Number

Let x_0 be a real number and let a be a positive real number less than 1. Now consider the real number x between $x_0 - a$ and $x_0 + a$ such that $x_0 - a < x < x_0 + a$.

In other words x is an element of the interval $(x_0 - a,\ x_0 + a)$. This interval is called the a-neighborhood of x_0.



For example, let us take $x_0 = 5$ and $a = 0.1$. The 0.1-neighborhood of 5 is the interval $(5 - 0.1,\ 5 + 0.1) = (4.9, 5.1)$.

Check Yourself 3

1. Write the 0.01-neighborhood of 7 as an interval and show it on a number line.
2. Write the 0.2-neighborhood of $\frac{5}{2}$ as an interval and show it on a number line.

2. ε-neighborhood of a Number

ε is a Greek letter, pronounced 'epsilon'. 'ε-neighborhood' is read as 'epsilon-neighborhood'.

Now let x be an element of the ε-neighborhood of x_0:

$$x \in (x_0 - \varepsilon,\ x_0 + \varepsilon).$$

We can write the ε-neighborhood of x_0 as an inequality: $x_0 - \varepsilon < x < x_0 + \varepsilon$.

When we subtract x_0 from all parts of this inequality we get: $-\varepsilon < x - x_0 < \varepsilon$.

We can write this inequality as an absolute value: $|x - x_0| < \varepsilon$.

For example, consider $\quad x \in (3.8,\ 4.2)$

$$x \in (4 - 0.2,\ 4 + 0.2)$$

$$(4 - 0.2) < x < (4 + 0.2)$$

$$-0.2 < x - 4 < 0.2$$

$$|x - 4| < 0.2.$$

This means that x is a real number in the 0.2-neighborhood of 4.

3. Limit of a Function

Definition

limit of a function

Let $f: \mathbb{R} \to \mathbb{R}$ be a function and let x_0 and L be real numbers.

Given any ε about L if there exists a δ about x_0 such that for all x,

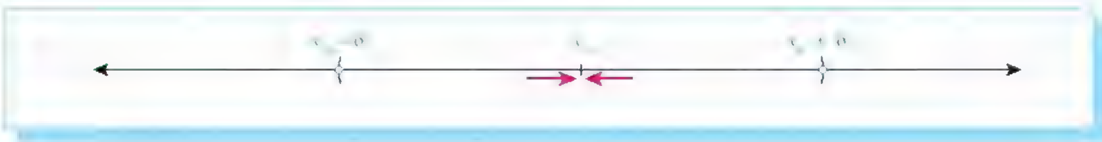
$$|x - x_0| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$$

then the limit of f as x approaches the point x_0 is L, i.e. $\lim\limits_{x \to x_0} f(x) = L$.

> δ is a Greek letter, pronounced 'delta'. 'δ-neighborhood' is read as 'delta-neighborhood'.

Let us look at each part of the statement in turn.

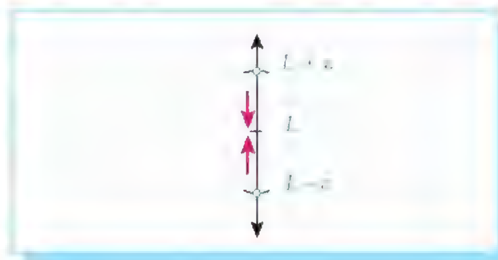
$$|x - x_0| < \delta$$
$$-\delta < x - x_0 < \delta$$
$$x_0 - \delta < x < x_0 + \delta$$
$$x \in (x_0 - \delta,\, x_0 + \delta), \text{ i.e. } x \text{ is in the } \delta\text{-neighborhood of } x_0.$$



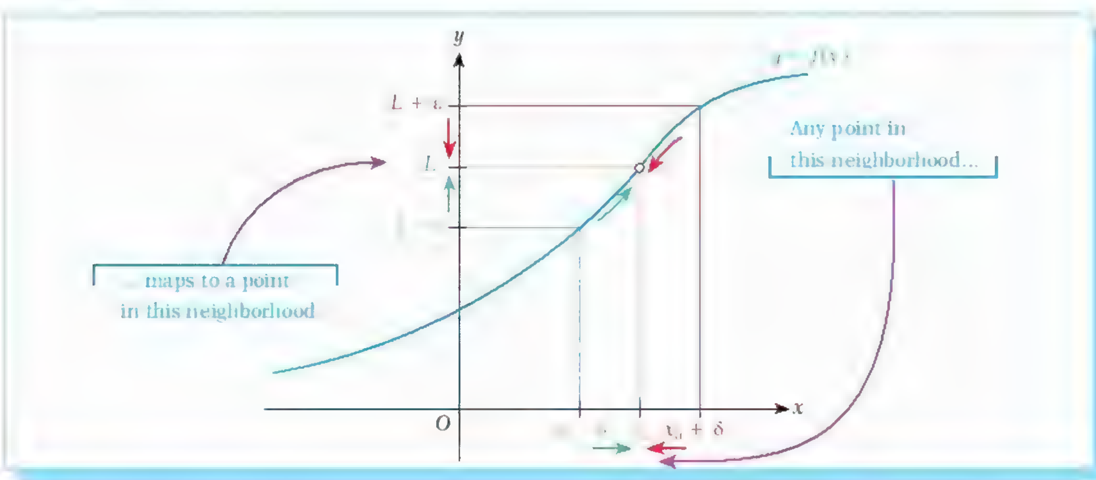
Similarly, $|f(x) - L| < \varepsilon$

$$-\varepsilon < f(x) - L < \varepsilon$$
$$L - \varepsilon < f(x) < L + \varepsilon$$
$$f(x) \in (L - \varepsilon,\, L + \varepsilon)$$

i.e. $f(x)$ is in the ε-neighborhood of L.



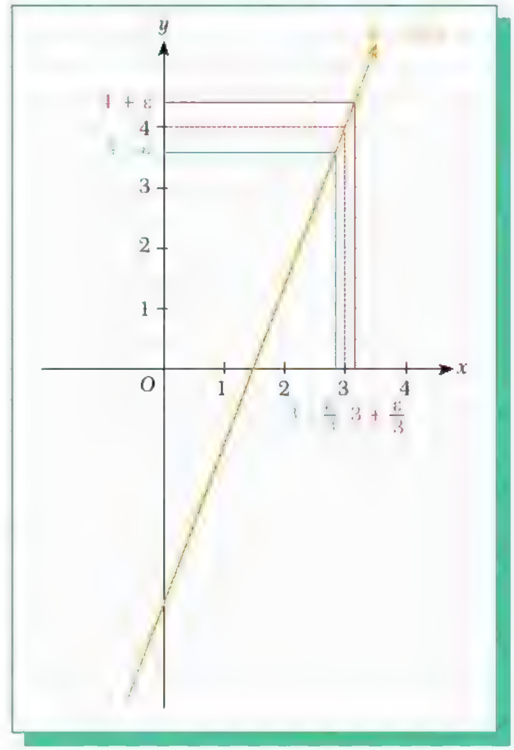
In other words, $\lim\limits_{x \to x_0} f(x) = L$ means that for any small number ε there is another small number δ such that any point in the δ-neighborhood of x corresponds to a point in the ε-neighborhood of L.



EXAMPLE 5 Show that $\lim\limits_{x\to 3}(3x-5)=4$ by using the definition of limit.

Solution We will use the definition of limit with $x_0 = 3$, $f(x) = 3x - 5$ and $L = 4$.

To satisfy the definition, we need to show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x, $|x-3| < \delta \Rightarrow |(3x-5)-4| < \varepsilon$.



Let us rewrite the second part of this statement:

$$|(3x-5)-4| < \varepsilon$$
$$|3x-9| < \varepsilon$$
$$3 \cdot |x-3| < \varepsilon$$
$$|x-3| < \frac{\varepsilon}{3}.$$

All these inequalities are equivalent. Now we need to show that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x,

$$|x-3| < \delta \;\Rightarrow\; |(3x-5)-4| < \varepsilon.$$

This is true if we take $\delta = \frac{\varepsilon}{3}$. Of course $\delta = \frac{\varepsilon}{3}$ is not the only value of δ that will satisfy the definition. Any smaller positive δ will work as well. However, the definition of limit asks us to find just one δ that satisfies the statement, so this is enough.

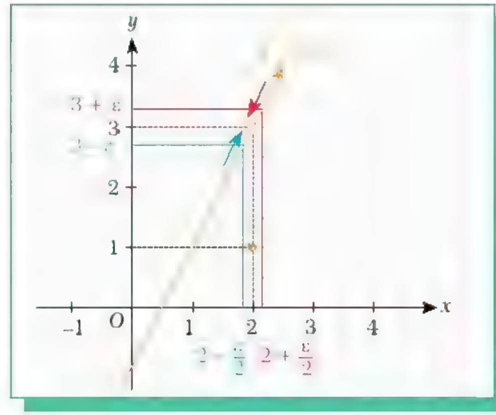
EXAMPLE 6 $f\colon \mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases} 2x-1 & \text{if } x\neq 2 \\ 1 & \text{if } x=2 \end{cases}$ is given. Show that $\lim\limits_{x\to 2} f(x)=3$ by using the definition of limit.

Solution To prove that $\lim\limits_{x\to 2} f(x)=3$ we need to show that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x-2| < \delta \Rightarrow |f(x)-3| < \varepsilon$.

Let us begin with the second part of the expression ($|f(x)-3| < \varepsilon$) and let us try to show that for all $\varepsilon > 0$ we can find a number δ.

$$|f(x)-3| < \varepsilon \;\Leftrightarrow\; |(2x-1)-3| < \varepsilon \Leftrightarrow |2x-4| < \varepsilon \Leftrightarrow |x-2| < \frac{\varepsilon}{2}$$

Now we have $|x - 2| < \frac{\varepsilon}{2}$ which is similar to the first part of the expression ($|x - 2| < \delta$).



Indeed, since $\frac{\varepsilon}{2}$ is a real number we can choose $\delta = \frac{\varepsilon}{2}$ and so $|x - 2| < \frac{\varepsilon}{2}$.

Now we have found that there is a real number δ which is equal to $\frac{\varepsilon}{2}$, so for any $\varepsilon > 0$ we can choose $\delta = \frac{\varepsilon}{2}$ for this function.

Moreover, we can say that for all $\varepsilon > 0$ there exists a $\delta = \frac{\varepsilon}{2} > 0$ and $|x - 2| < \delta \Rightarrow |f(x) - 3| < \varepsilon$.

For $x \in (2 - \delta, 2 + \delta) = (2 - \frac{\varepsilon}{2}, 2 + \frac{\varepsilon}{2})$, $f(x) \in (3 - \varepsilon, 3 + \varepsilon)$.

So as ε approaches zero, the point x approaches 2 and the value of $f(x)$ approaches 3.

Thus, $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2}(2x - 1) = 3$.

Check Yourself 4

Use the definition of limit to prove each statement and find the value of δ.

1. $\lim\limits_{x \to 1}(7x - 2) = 5$
2. $\lim\limits_{x \to -2}(4x + 3) = -5$

Answers

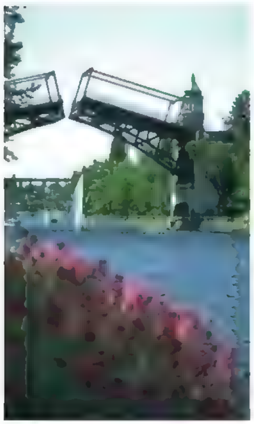
1. $\frac{\varepsilon}{7}$
2. $\frac{\varepsilon}{4}$

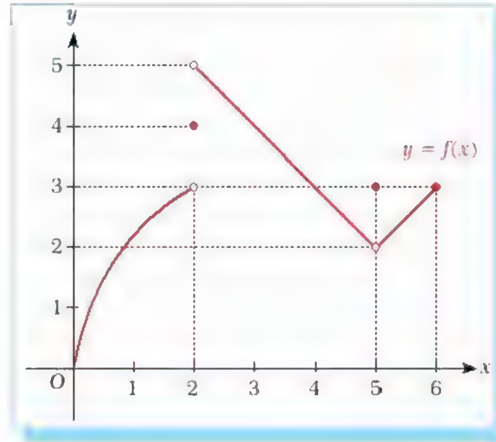
D. ONE-SIDED LIMITS

Notice that in limit notation, $x \to x_0$ and $x \to x_0^+$ mean different things. $x \to -x_0$ and $x \to x_0^-$ also have different meanings.

As we have already seen, sometimes the limit of a function can have two different values: one value when x approaches x_0 from the right, and another when x approaches x_0 from the left. When this happens, we call the limit of f as x approaches x_0 from the right the right-hand limit of f at x_0 and write it as $\lim\limits_{x \to x_0^+} f(x)$.

We call the limit of f as x approaches x_0 from the left the left-hand limit of f at x_0 and write it as $\lim\limits_{x \to x_0^-} f(x)$.





For example, consider the function $y = f(x)$ shown opposite. Let us find the left-hand and right-hand limits at the points 2, 5 and 6:

a. $\lim\limits_{x \to 2^-} f(x) = 3$ and $\lim\limits_{x \to 2^+} f(x) = 5$

b. $\lim\limits_{x \to 5^-} f(x) = 2$ and $\lim\limits_{x \to 5^+} f(x) = 2$

c. $\lim\limits_{x \to 6^-} f(x) = 3$ and $\lim\limits_{x \to 6^+} f(x)$ does not exist, since f is not defined for $x > 6$.

In other words the left-hand and right-hand limits as $x \to 2$ are different; as $x \to 5$ the left- and right-hand limits are the same, and as $x \to 6$ only the left-hand limit exists.

Definition

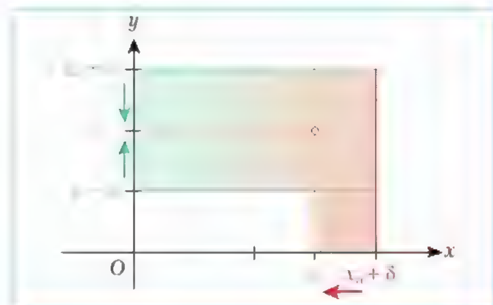
one-sided limits

1. The limit of the function $f(x)$ as x approaches x_0 from the right equals L if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x,

$$x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \delta,$$

i.e. for all x in the interval $(x_0, x_0 + \delta)$,

$$f(x) \in (L - \varepsilon, L + \varepsilon).$$

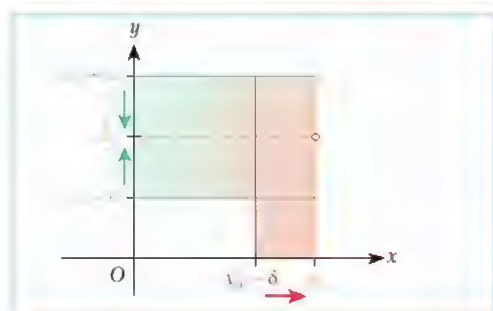


2. The limit of the function $f(x)$ as x approaches x_0 from the left equals L if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x,

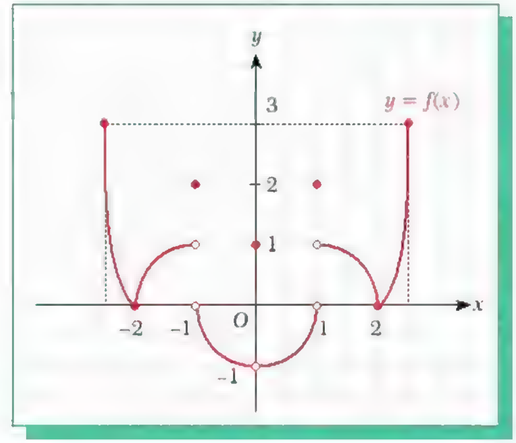
$$x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon,$$ i.e.

for all x in the interval $(x_0 - \delta, x_0)$,

$$f(x) \in (L - \varepsilon, L + \varepsilon).$$



EXAMPLE 7 Find the one-sided limits at all the integer values of the graph of $f:\left[-\frac{5}{2}, \frac{5}{2}\right] \to \mathbb{R}$ shown in the figure.



Solution The integer values are {–2, –1, 0, 1, 2}, so we need to inspect these points.

a. $\lim\limits_{x\to 2^+} f(x) = 0$, $\lim\limits_{x\to 2^-} f(x) = 0$

b. $\lim\limits_{x\to 1^+} f(x) = 1$, $\lim\limits_{x\to 1^-} f(x) = 0$

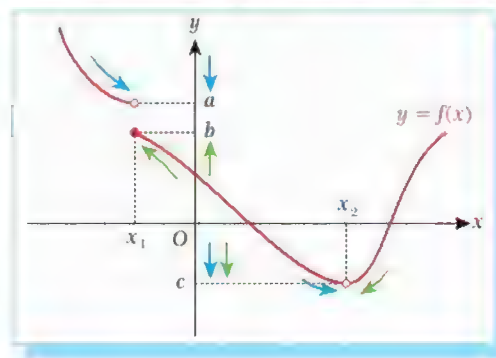
c. $\lim\limits_{x\to 0^+} f(x) = -1$, $\lim\limits_{x\to 0^-} f(x) = -1$

d. $\lim\limits_{x\to -1^+} f(x) = 0$, $\lim\limits_{x\to -1^-} f(x) = 1$

e. $\lim\limits_{x\to -2^+} f(x) = 0$, $\lim\limits_{x\to -2^-} f(x) = 0$.

Definition existence of a limit

The limit of a function $f(x)$ at a point x_0 exists if and only if the right-hand and left-hand limits at x_0 exist and are equal.



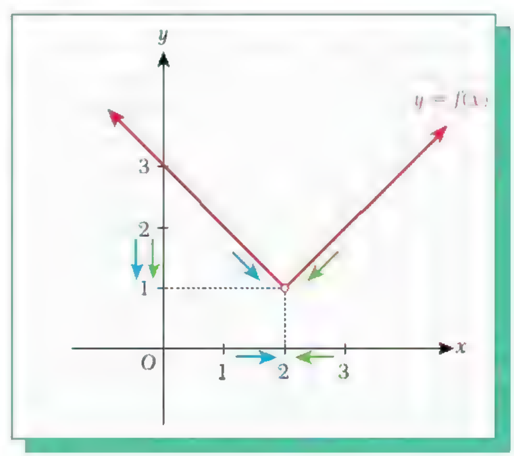
In other words,

$$\lim_{x\to x_0} f(x) = L \iff \lim_{x\to x_0^+} f(x) = L \text{ and } \lim_{x\to x_0^-} f(x) = L.$$

The left-hand limit and right-hand limit of a function are also called the one-sided limits of the function.

For example, in the figure the function f has a limit at point x_2, but it has no limit at point x_1 because the left-hand and right-hand limits at x_1 are different.

EXAMPLE 8 $f: \mathbb{R} - \{2\} \to \mathbb{R}$, $f(x) = \begin{cases} x-1 & \text{if } x > 2 \\ -x+3 & \text{if } x < 2 \end{cases}$ is given. Find $\lim\limits_{x\to 2} f(x)$.



Solution As we can see from the graph, the point $x_0 = 2$ is the crucial point of $f(x)$.

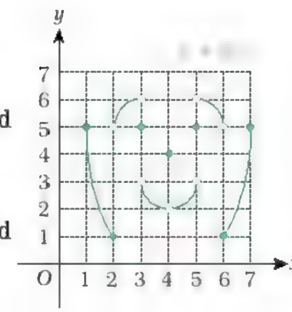
Therefore, let us examine the one-sided limits at this point.

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}(x-1) = 2-1 = 1$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-}(-x+3) = -2+3 = 1$

Since $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x) = 1$, they exist and are equal, and so $\lim\limits_{x\to 2} f(x) = 1$.

Check Yourself 5



1. The graph of $f: [1, 7] \to \mathbb{R}$, $f(x)$ is shown in the figure. Find the one-sided limits at all integer values of the domain and find at which point f has a limit.

2. Given the function $f: \mathbb{R} - \{1\} \to \mathbb{R}$, $f(x) = \begin{cases} x^2+1 & \text{if } x > 1 \\ x-1 & \text{if } x < 1 \end{cases}$, find the limit of $f(x)$ at the point $x_0 = 1$.

3. The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x & \text{if } -1 \le x < 0 \text{ or } 0 < x \le 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x < -1 \text{ or } x > 1 \end{cases}$ is given.
   a. Find the one-sided limits at the points –1, 0, and 1.
   b. At which point does the limit of $f(x)$ exist?

Answers

2. does not exist

E. LIMITS OF SPECIAL FUNCTIONS

We have now learnt the definition and basic concepts of the limit of a function, and studied one-sided limits. In this section we will look at the limit of some special functions: the absolute value function, the sign function and the floor function.

We know that at a given point, the limit of a function exists if the right-hand limit and the left-hand limit exist and are equal.

We can evaluate a limit of an absolute value, sign or floor function at a point by treating the function as a piecewise function and checking the one-sided limits at the point. If the two limits exist and are equal, then we can say that a limit exists at the given point.

EXAMPLE 9 $f: \mathbb{R} - \{2\} \to \mathbb{R}$, $f(x) = \dfrac{|x-2|}{x-2} + x + 3$ is given. Find $\lim\limits_{x \to 2^+} f(x)$ and $\lim\limits_{x \to 2^-} f(x)$.

Solution Since the function f involves the absolute value expression $|x-2|$, $x_0 = 2$ is a crucial point for f.

Let us begin by writing the function as a piecewise function.

If $x > 2$, $x - 2 > 0$ and so $|x-2| = x - 2$. Therefore

$$f(x) = \frac{|x-2|}{x-2} + x + 3 = \frac{x-2}{x-2} + x + 3 = 1 + x + 3 = x + 4.$$

Similarly, if $x < 2$,

$x - 2 < 0$ and $|x-2| = -(x-2)$ and so

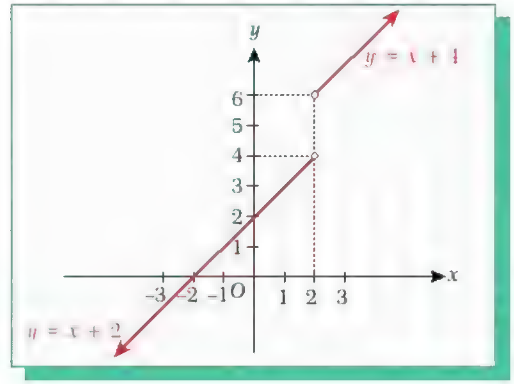
$$f(x) = \frac{|x-2|}{x-2} + x + 3 = \frac{-(x-2)}{x-2} + x + 3$$
$$= -1 + x + 3 = x + 2.$$

In conclusion, $f(x) = \begin{cases} x+4 & \text{if } x > 2 \\ x+2 & \text{if } x < 2. \end{cases}$

So the limits are

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x+4) = 2+4 = 6$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x+2) = 2+2 = 4.$$



Remark

When we evaluate the right-hand and left-hand limits of special functions it is sometimes useful to define a positive real number h and look at what happens as h approaches zero.

For $h \in \mathbb{R}^+$ and $h \to 0$, the following statements are true:

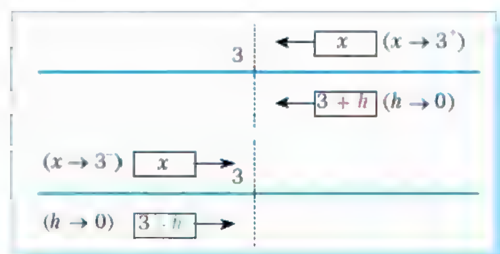
$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h) \text{ and}$$

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h).$$

For example, when $h \in \mathbb{R}^+$ and $h \to 0$,

$$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h) \text{ and}$$

$$\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h).$$



EXAMPLE 10 Given $f(x) = \operatorname{sgn}(4-x)$, find $\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4^-} f(x)$ and decide whether $\lim_{x\to 4} f(x)$ exists or not.

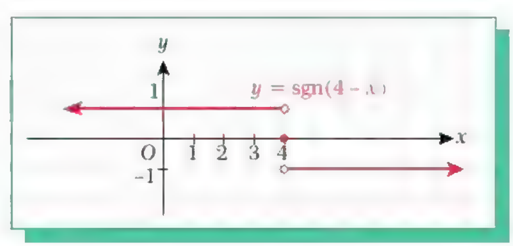
Solution 1 $x_0 = 4$ is a crucial point.

When $x > 4$ $(x \to 4^+)$, $4 - x$ is negative and so $\operatorname{sgn}(4-x) = -1$. So

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \operatorname{sgn}(4-x) = -1.$$

When $x < 4$ $(x \to 4^-)$, $4 - x$ is positive and so $\operatorname{sgn}(4-x) = 1$. So

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \operatorname{sgn}(4-x) = 1.$$



Since the left-hand and right-hand limits are not equal, $\lim_{x\to 4} f(x)$ does not exist.

Solution 2 Let $h > 0$ be a very small real number. Instead of writing $x \to 4^+$, we can write $x = 4 + h$ and consider $h \to 0$:

$\lim\limits_{x\to 4^+} \operatorname{sgn}(4 - x) = \lim\limits_{h\to 0} \operatorname{sgn}(4 - (4 + h)) = \lim\limits_{h\to 0} \operatorname{sgn}(-h)$.

Since h is a very small positive number, $-h$ is negative and so $\operatorname{sgn}(-h) = -1$,

i.e. $\lim\limits_{h\to 0} \operatorname{sgn}(-h) = -1$. This is the right-hand limit.

Similarly, $\lim\limits_{x\to 4^-} \operatorname{sgn}(4 - x) = \lim\limits_{h\to 0} \operatorname{sgn}(4 - (4 - h)) = \lim\limits_{h\to 0} \operatorname{sgn}(h) = 1$.

Since h is *a* positive number, $\operatorname{sgn}(h) = 1$. This is the left-hand limit.

As before, $\lim\limits_{x\to 4^+} f(x) \neq \lim\limits_{x\to 4^-} f(x)$ and so $\lim\limits_{x\to 4} f(x)$ does not exist.

EXAMPLE 11 The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = [\![5x + 4]\!]$ is given. Find $\lim\limits_{x\to 2} f(x)$.

Solution To find the limit as x approaches 2, we have to check the right-hand and left-hand limits:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} [\![5x + 4]\!] = \lim_{h\to 0} [\![5(2 + h) + 4]\!] = \lim_{h\to 0} [\![14 + 5h]\!] = 14$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} [\![5x + 4]\!] = \lim_{h\to 0} [\![5(2 - h) + 4]\!] = \lim_{h\to 0} [\![14 - 5h]\!] = 13.$$

Notice that since h is a very small positive number, $14 - 5h < 14$ and so $[\![14 - 5h]\!] = 13$.

We can see that $\lim\limits_{x\to 2^+} f(x) \neq \lim\limits_{x\to 2^-} f(x)$.

So $\lim\limits_{x\to 2} f(x)$ does not exist.

EXAMPLE 12 Find $\lim\limits_{x\to 1^-} \dfrac{|x-1|}{|1-x|}$.

Solution $x \to 1^-$ means x is less than 1. Therefore, $x - 1 < 0$ so $|x - 1| = -(x - 1) = 1 - x$, and $1 - x > 0$ so $|1 - x| = 1 - x$.

So $\lim\limits_{x\to 1^-} \dfrac{|x-1|}{|1-x|} = \lim\limits_{x\to 1^-} \dfrac{1-x}{1-x} = \lim\limits_{x\to 1^-} 1 = 1$.

EXAMPLE 13 Find $\lim_{x\to 1^-} \operatorname{sgn}(\frac{x^2-1}{x+2})$.

Solution Let us make a sign table of $\frac{x^2-1}{x+2}$.

x		-2		-1		1	
$\frac{x^2-1}{x+2}$	$-$	$\circ$	$+$	$\circ$	$-$	$\circ$	$+$

$x \to 1^-$ means that x approaches 1 from the left-hand side, and in this interval $\frac{x^2-1}{x+2}$ is negative.

So $\lim_{x\to 1^-} \operatorname{sgn}(\frac{x^2-1}{x+2}) = -1$.

EXAMPLE 14 Find $\lim_{x\to 2^-} [\![2x+1]\!]$.

Solution Let $h > 0$ be a very small real number.

Then $\lim_{x\to 2^-}[\![2x+1]\!] = \lim_{h\to 0}[\![2(2-h)+1]\!] = \lim_{h\to 0}[\![5-2h]\!] = 4$.

EXAMPLE 15 Find the right-hand and the left-hand limits of the function $f(x) = \frac{|x-3|+\operatorname{sgn}(x-2)+x}{[\![-2x]\!]}$ as x approaches 2.

Solution Let $h > 0$ be a very small real number.

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 2^+} \frac{|x-3|+\operatorname{sgn}(x-2)+x}{[\![-2x]\!]} = \lim_{h\to 0} \frac{|2+h-3|+|\operatorname{sgn}(2+h-2)|+2+h}{[\![-2(2+h)]\!]}$$

$$= \lim_{h\to 0} \frac{|-1+h|+\operatorname{sgn}(h)+2+h}{[\![-4-2h]\!]}$$

$$= \lim_{h\to 0} \frac{1+1+2}{-5} = -\frac{4}{5}.$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 2^-} \frac{|x-3|+\operatorname{sgn}(x-2)+x}{[\![-2x]\!]} = \lim_{h\to 0} \frac{|2-h-3|+|\operatorname{sgn}(2-h-2)|+2-h}{[\![-2(2-h)]\!]}$$

$$= \lim_{h\to 0} \frac{|-1-h|+\operatorname{sgn}(-h)+2-h}{[\![-4+2h]\!]} = \lim_{h\to 0} \frac{1-1+2}{-4} = -\frac{1}{2}.$$

Check Yourself 6

Evaluate the limits.

1. $\lim_{x\to -3}(x^2\,|x+1|)$
2. $\lim_{x\to -1}\frac{x+|x|}{x}$
3. $\lim_{x\to 2^-}\frac{|x^2-4|}{x-2}$
4. $\lim_{x\to 1^-}\frac{|x^2-3x+2|}{x-1}$
5. $\lim_{x\to 2^+}\frac{|2-x|}{\operatorname{sgn}(x-2)}$
6. $\lim_{x\to 3^+}\operatorname{sgn}(3-x)-x$
7. $\lim_{x\to 0^+}\frac{|x|+2}{\operatorname{sgn}(x^2-2)}$
8. $\lim_{x\to 1^+}\frac{1-x}{\operatorname{sgn}(x^2-1)}$
9. $\lim_{x\to 2^+}[\![2x+1]\!]$
10. $\lim_{x\to 3^-}[\![\frac{5x+1}{16}]\!]$
11. $\lim_{x\to 2}([\![x]\!]+x+2)$
12. $\lim_{x\to 4^+}[\![\frac{8x+1}{3}]\!]$
13. $\lim_{x\to 0^-}([\![2007x]\!]-1)$
14. $\lim_{x\to 3^-}[\![2[\![x]\!]]\!]$
15. $\lim_{x\to 4}\sqrt{[\![3x+4]\!]-x}$

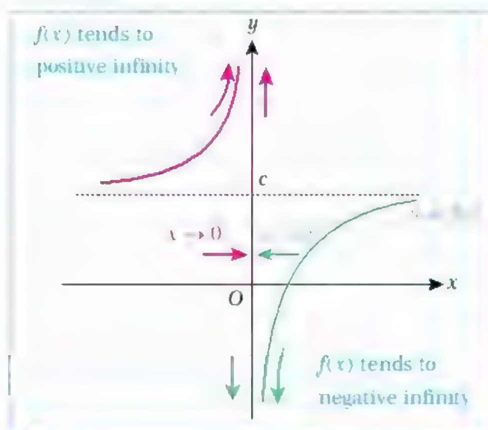
Answers

1. 18 2. 0 3. –4 4. –1 5. 0 6. –4 7. –2 8. 0 9. 5 10. 0 11. does not exist 12. 11 13. –2 14. 4 15. does not exist

F. LIMITS INVOLVING INFINITY

In this section, we will use the concept of infinity. Infinity is not a real number, but we can use it to describe a graph which continues without end in a positive or negative direction.

For example, consider the function $f:\mathbb{R}-\{0\}\to\mathbb{R}-\{c\}$ graphed in the figure. How can we evaluate $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$?



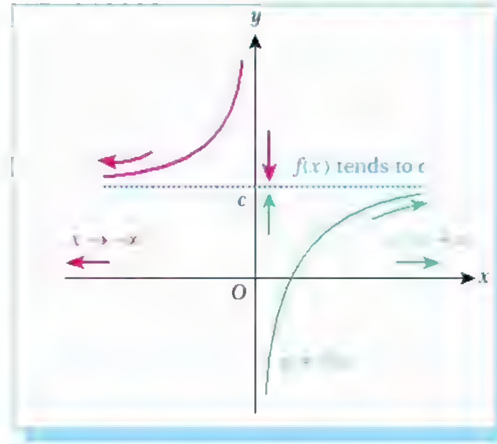
As we can see in the graph, as we approach zero from the left-hand side the value of $f(x)$ gets larger and larger. In other words, for any chosen number M, we can always find an x closer to zero on the left such that $f(x) > M$. We say that $f(x)$ tends to (i.e. moves in the direction of) positive infinity, and write

$$\lim_{x\to 0^-} f(x) = +\infty.$$

Similarly, when we approach zero from the right-hand side the value of $f(x)$ gets smaller and smaller. In other words, for any chosen number $-M$, we can always find an x closer to zero on the right such that $f(x) < -M$. We say that $f(x)$ tends to negative infinity:

$$\lim_{x\to 0^+} f(x) = -\infty.$$

In both cases, $f(x)$ has an infinite limit as x approaches a real number. Now, what about $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$?



As we can see in the graph, as the value of x increases, the value of $f(x)$ approaches the number c. In other words, for any chosen $\varepsilon > 0$, we can find a number M such that for all $x > M$, the value of $f(x)$ will be in the ε-neighborhood of c, i.e. $f(x)$ is getting closer and closer to c. So we can write

$$\lim_{x\to+\infty} f(x) = c.$$

Similarly, as x gets smaller and smaller, the value of $f(x)$ also approaches the number c as shown in the figure. For chosen any $\varepsilon > 0$, we can find a number N such that for all $x < N$, the value of $f(x)$ will be in the ε-neighborhood of c, i.e. $f(x)$ approaches the number c while x approaches negative infinity. So we can write

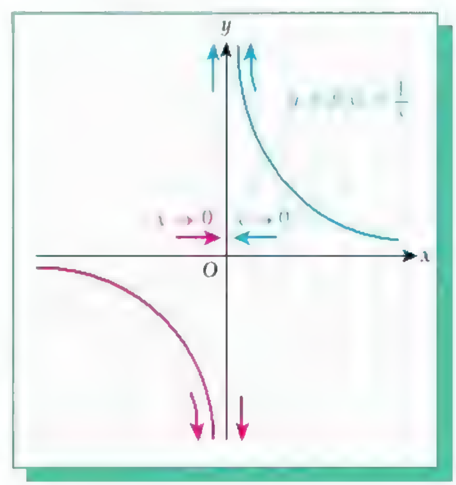
$$\lim_{x\to-\infty} f(x) = c.$$

In both cases, as x approaches negative or positive infinity, $f(x)$ approaches a real number.

EXAMPLE 16 $f: \mathbb{R} - \{0\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x}$ is given. Find the limits.

a. $\lim\limits_{x\to0^+} f(x)$ b. $\lim\limits_{x\to0^-} f(x)$ c. $\lim\limits_{x\to+\infty} f(x)$ d. $\lim\limits_{x\to-\infty} f(x)$

Solution First, let us graph $f(x) = \dfrac{1}{x}$ to study the limits of the function.



a. As x approaches zero from the right-hand side, $f(x)$ approaches $+\infty$:

$$\lim_{x\to0^+} \frac{1}{x} = +\infty.$$

For example, when we choose a positive number which is very close to zero such as $x = 0.0001$, we get a large positive number: $\dfrac{1}{x} = 10000$.

b. As x approaches zero from the left-hand side, $f(x)$ approaches $-\infty$:

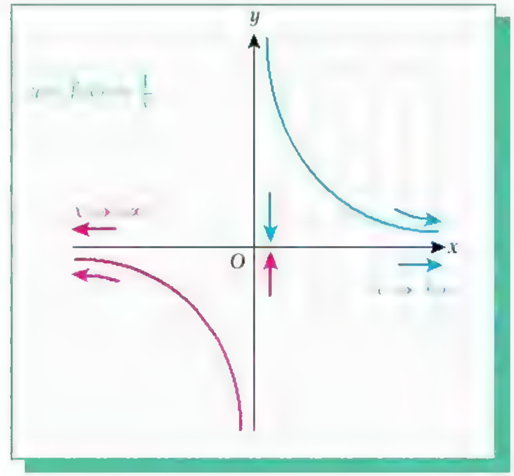
$$\lim_{x\to0^-} \frac{1}{x} = -\infty.$$

For example, when we choose a negative number which is very close to zero such as $x = -0.00001$, we get a large negative number: $\dfrac{1}{x} = -100000$.

c. As x approaches $+\infty$, $f(x)$ approaches zero:

$$\lim_{x \to +\infty} \frac{1}{x} = 0.$$

For example, when we choose a large positive value of x such as $x = 1000000$, we get a small positive number close to zero: $\frac{1}{x} = 0.000001$.



d. As x approaches $-\infty$, $f(x)$ approaches zero:

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

For example, when we choose a large negative value of x such as $x = -10000000$, we get a small negative number close to zero:

$$\frac{1}{x} = -0.0000001.$$

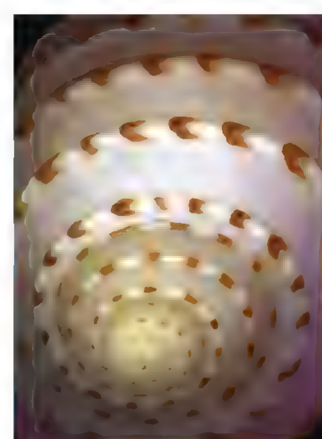
As a consequence of results a and b, notice that $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist.

EXAMPLE 17 Find the limit of the function $f\colon \mathbb{R} - \{0\} \to \mathbb{R}$, $f(x) = \dfrac{2x^2 - x}{x^2 + 4}$ as x approaches $+\infty$.

Solution

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{2x^2 - x}{x^2 + 4} = \lim_{x \to +\infty} \frac{x^2\left(2 - \frac{1}{x}\right)}{x^2\left(1 + \frac{4}{x^2}\right)} = \lim_{x \to +\infty} \frac{\left(2 - \frac{1}{x}\right)}{\left(1 + \frac{4}{x^2}\right)}$$

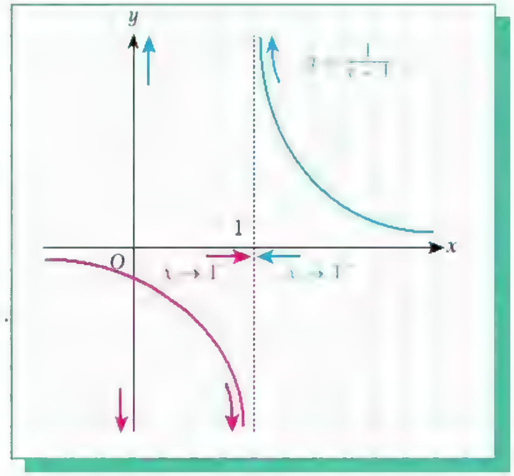
As $x \to +\infty$ we have $\frac{1}{x} \to 0$ and $\frac{4}{x^2} \to 0$, so $\lim\limits_{x \to +\infty} \dfrac{\left(2 - \overset{0}{\cancel{\frac{1}{x}}}\right)}{\left(1 + \underset{0}{\cancel{\frac{4}{x^2}}}\right)} = 2.$



EXAMPLE 18 Find the limit of the function $f(x)=\dfrac{1}{x-1}$ as $x \to 1$.

Solution The function f is not defined at $x_0 = 1$ and so $x_0 = 1$ is a crucial point.

So let us check the right-hand and left-hand limits. Let $h > 0$ be a very small number, then



$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{1}{x-1}=\lim_{h\to 0}\frac{1}{(1+h)-1}=\lim_{h\to 0}\frac{1}{h}=+\infty,$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{1}{x-1}=\lim_{h\to 0}\frac{1}{(1-h)-1}=\lim_{h\to 0}\frac{1}{-h}=-\infty.$$

The right-hand and left-hand limits are not equal, so $\lim\limits_{x\to 1} f(x)$ does not exist.

Remark

For some functions f: $\mathbb{R} \to \mathbb{R}$, the following infinite limits are possible:

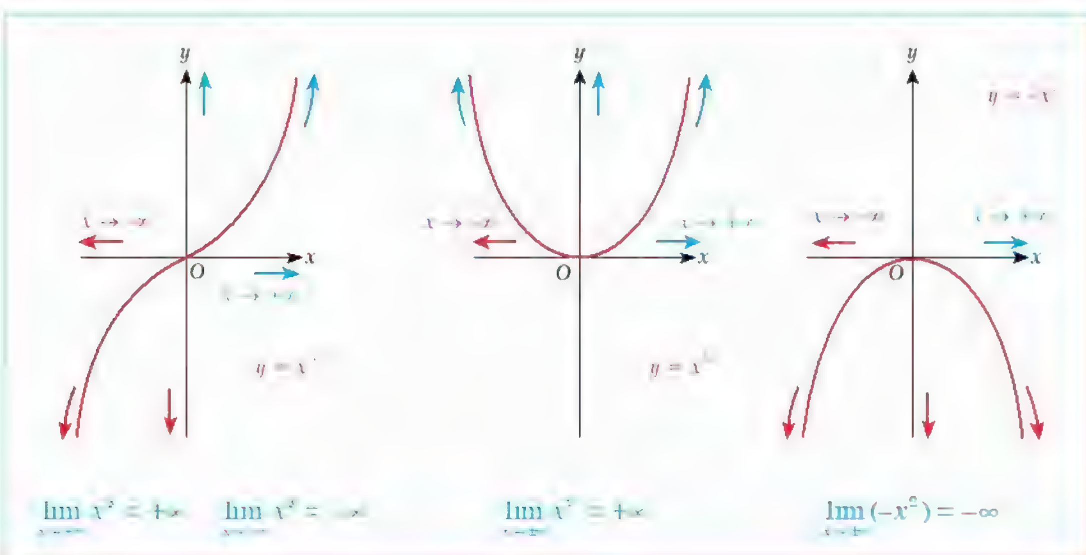
a. $\lim\limits_{x\to +\infty} f(x)=+\infty$

b. $\lim\limits_{x\to +\infty} f(x)=-\infty$

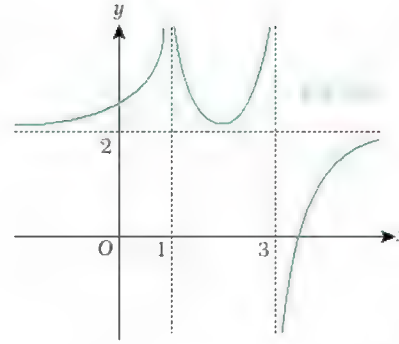
c. $\lim\limits_{x\to -\infty} f(x)=+\infty$

d. $\lim\limits_{x\to -\infty} f(x)=-\infty.$

Look at the examples.



Check Yourself 7



1. The graph of a function $f(x)$ is shown opposite. Find each limit.

   a. $\lim\limits_{x\to+\infty} f(x)$ b. $\lim\limits_{x\to-\infty} f(x)$

   c. $\lim\limits_{x\to1} f(x)$ d. $\lim\limits_{x\to3} f(x)$

2. Calculate the limits.

   a. $\lim\limits_{x\to+\infty} \dfrac{2x+3}{5x+4}$ b. $\lim\limits_{x\to+\infty} \dfrac{x^3+1}{x^4}$

   c. $\lim\limits_{x\to+\infty} \dfrac{x^2-2x+3}{2x^2+3x-1}$ d. $\lim\limits_{x\to-\infty} \dfrac{|x|}{|x|+1}$ e. $\lim\limits_{x\to-\infty} \dfrac{1-x^2}{1+3x^2}$ f. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2-2}}{4x-2}$

3. Calculate the limits.

   a. $\lim\limits_{x\to0} \dfrac{1}{x^2}$ b. $\lim\limits_{x\to1^-} \dfrac{1}{x-1}$ c. $\lim\limits_{x\to2^+} \dfrac{1}{x^2-4}$ d. $\lim\limits_{x\to e} \dfrac{1}{1-\ln x}$ e. $\lim\limits_{x\to2} \dfrac{5x-1}{|2-x|}$

4. Calculate the limits.

   a. $\lim\limits_{x\to-\infty} (x^5+x^4+x^3)$ b. $\lim\limits_{x\to\infty}(1-x-x^2)$ c. $\lim\limits_{x\to-\infty} (2x-\dfrac{3}{x})$

Answers

1\. a. 2 b. 2 c. $+\infty$ d. does not exist 2. a. $\dfrac{2}{5}$ b. 0 c. $\dfrac{1}{2}$ d. 1 e. $-\dfrac{1}{3}$ f. $-\dfrac{1}{4}$ 3. a. $+\infty$ b. $-\infty$ c. $+\infty$ d. does not exist e. $+\infty$ 4. a. $-\infty$ b. $-\infty$ c. $-\infty$

Theorem

Let $f(x)$ and $g(x)$ be functions such that $\lim\limits_{x\to x_0} f(x) = a$ and $\lim\limits_{x\to x_0} g(x) = b$. Then

a. $\lim\limits_{x\to x_0} [f(x)+g(x)] = a+b$

b. $\lim\limits_{x\to x_0} [f(x)-g(x)] = a-b$

c. $\lim\limits_{x\to x_0} [f(x)\cdot g(x)] = a\cdot b$

d. $\lim\limits_{x\to x_0} [\dfrac{f(x)}{g(x)}] = \dfrac{a}{b}$ $(b \neq 0)$

e. $\lim\limits_{x\to x_0} k\cdot f(x) = k\cdot a$ $(k \in \mathbb{R})$.

EXAMPLE 19 Given $f(x) = 5$ and $g(x) = \dfrac{1}{x}$ $(x \neq 0)$, evaluate the limits.

a. $\lim\limits_{x\to-2} [f(x)+g(x)]$ b. $\lim\limits_{x\to+\infty} [f(x)\cdot g(x)]$ c. $\lim\limits_{x\to-3} [\dfrac{f(x)}{g(x)}]$

Solution a. $\lim_{x\to -2}[f(x)+g(x)] = \lim_{x\to -2} f(x) + \lim_{x\to -2} g(x) = \lim_{x\to -2} 5 + \lim_{x\to -2} \frac{1}{x} = 5 + (-\frac{1}{2}) = \frac{9}{2}$

b. $\lim_{x\to +\infty}[f(x)\cdot g(x)] = \lim_{x\to +\infty} f(x) \cdot \lim_{x\to +\infty} g(x) = \lim_{x\to +\infty} 5 \cdot \lim_{x\to +\infty} \frac{1}{x} = 5 \cdot 0 = 0$

c. $\lim_{x\to -3}[\frac{f(x)}{g(x)}] = \lim_{x\to -3} \frac{5}{\frac{1}{x}} = \lim_{x\to -3} 5\cdot x = 5 \cdot \lim_{x\to -3} x = 5(-3) = -15$

EXAMPLE 20 $f\colon \mathbb{R} - \{0\} \to \mathbb{R},\ f(x) = \frac{1}{1+2^{2/x}}$ is given. Calculate the limits.

a. $\lim_{x\to 0^+} f(x)$ b. $\lim_{x\to 0^-} f(x)$ c. $\lim_{x\to 0} f(x)$

Solution a. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{1+2^{2/x}} = \frac{\lim_{x\to 0^+} 1}{\lim_{x\to 0^+} 1 + \lim_{x\to 0^+} 2^{2/x}}$

As $x \to 0^+$, $\frac{2}{x} \to +\infty$ and so $2^{2/x} \to +\infty$. So $\frac{\lim_{x\to 0^+} 1}{\lim_{x\to 0^+} 1 + \lim_{x\to 0^+} 2^{2/x}} = \frac{1}{1+\infty} = 0.$

b. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{1+2^{2/x}} = \frac{\lim_{x\to 0^-} 1}{\lim_{x\to 0^-} 1 + \lim_{x\to 0^-} 2^{2/x}}$

As $x \to 0^-$, $\frac{2}{x} \to -\infty$ and so $\frac{\lim_{x\to 0^-} 1}{\lim_{x\to 0^-} 1 + \lim_{x\to 0^-} 2^{2/x}} = \frac{1}{1+2^{-\infty}} = \frac{1}{1+\frac{1}{2^{\infty}}} = \frac{1}{1+0} = 1.$

c. Since $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, $\lim_{x\to 0} f(x)$ does not exist.

Check Yourself 8

1. Given $\lim_{x\to a} f(x) = 2$, find $\lim_{x\to a} \frac{f^2(x) - f(x)}{f(x)+3}$.

2. $f(x) = \sin x + 2$ and $g(x) = \cos 2x$ are given. Find $\lim_{x\to \frac{\pi}{2}} \frac{f(x)\cdot g(x) + f(x)}{\left(\frac{g}{f}\right)(x)}$.

3. Calculate each limit by using the limit combination theorem.

a. $\lim_{x\to 1} \frac{x-2}{x^2+4}$ b. $\lim_{x\to -\infty} \frac{1+\frac{2}{x}}{5+\frac{4}{x}}$ c. $\lim_{x\to 1} (3x+1)^2$ d. $\lim_{x\to -1} (6x^3+4x^2-3)$

Answers

1. $\frac{2}{5}$ 2. 0 3. a. $-\frac{1}{5}$ b. $\frac{1}{5}$ c. 16 d. –5

EXERCISES 2.1

A. Limit of a Polynomial Function

In questions 1-7, calculate the limits.

1. $\lim_{x \to 5}(3x+2)$

2. $\lim_{x \to -4}(x^2-2x)$

3. $\lim_{t \to 2} 3(2t-1)(t+1)$

4. $\lim_{x \to 4} 17$

5. $\lim_{x \to -2}(2k+1)$

6. $\lim_{x \to a}(2x+1) \cdot x^2$

7. $\lim_{x \to b}(x^2-2b+1)$

B. Limits on a Graph

In questions 8-10, draw the graph of each function and calculate the given limits.

8. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 4x - 7$, $\lim_{x \to 3} f(x)$

9. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^2 & \text{if} \quad x > 0 \\ 3 & \text{if} \quad x = 0 \\ -x^2 & \text{if} \quad x < 0 \end{cases}$

   a. $\lim_{x \to 0} f(x)$ b. $\lim_{x \to 2} f(x)$ c. $\lim_{x \to -2} f(x)$

10. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 2 & \text{if} \quad x > -3 \\ 1 & \text{if} \quad x = -3 \\ x+3 & \text{if} \quad x < -3 \end{cases}$

   a. $\lim_{x \to -5} f(x)$ b. $\lim_{x \to -3} f(x)$ c. $\lim_{x \to -1} f(x)$

C. Definition of Limit

11. Write the 0.3-neighborhood of 12 as an interval.

12. Write the 0.1-neighborhood of –5 as an interval and show it on a number line.

13. Write the 0.01-neighborhood of 4 as an interval.

14. Write the 0.05-neighborhood of 5 as an interval.

15. Use the definition of the limit of a function to prove each statement.

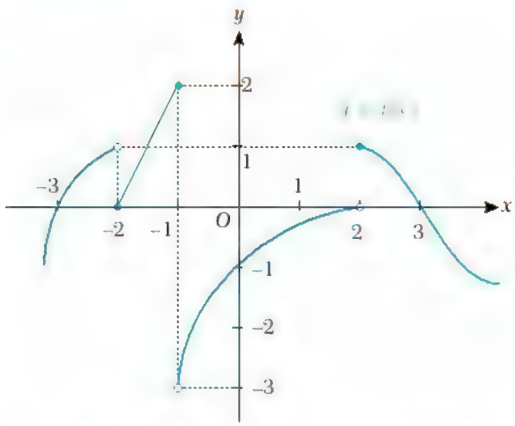
   a. $\lim_{x \to 5} 2x = 10$

   b. $\lim_{x \to -\frac{1}{2}} (2x+3) = 2$

   c. $\lim_{x \to 3} 5 = 5$

D. One-Sided Limits

16.



The figure shows the graph of a function $f(x)$. Find each limit.

a. $\lim\limits_{x \to -1^+} f(x)$

b. $\lim\limits_{x \to 2^+} f(x)$

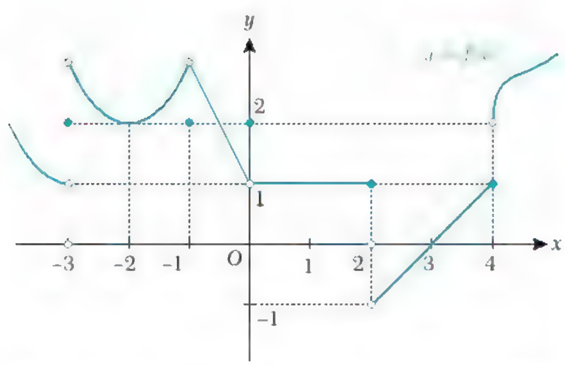
c. $\lim\limits_{x \to 0^-} f(x)$

d. $\lim\limits_{x \to -2^+} f(x)$

e. $\lim\limits_{x \to -2^-} f(x)$

f. $\lim\limits_{x \to -1^-} f(x)$

17.



The graph of a function f is shown in the figure. For which integer values of p in the interval $(-\frac{7}{2}, \frac{9}{2})$ does $\lim\limits_{x \to p} f(x)$ exist?

18. The function f: $\mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x & \text{if } x > 1 \\ 2 & \text{if } x = 1 \text{ or } x = -1 \\ -x^2 + 1 & \text{if } -1 < x < 1 \\ -x & \text{if } x < -1 \end{cases}$$

is given.

a. Find the one-sided limits as $x \to -1$, $x \to 0$ and $x \to 1$.

b. Find at which point(s) the limit does not exist.

19. The function f: $\mathbb{R} \to \mathbb{R}$,

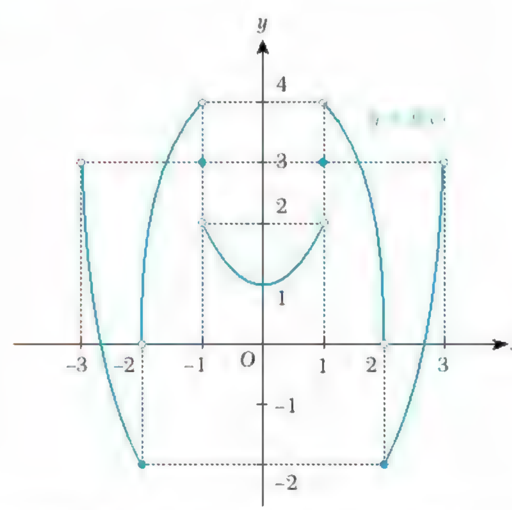
$$f(x) = \begin{cases} -3 & \text{if } x > 1 \\ 2 & \text{if } x = 1 \text{ or } x \le -2 \\ x^2 - 4 & \text{if } -2 < x < 1 \end{cases}$$

is given.

a. Find the limits as $x \to 0$ and $x \to 1$.

b. Find at which point(s) the limit does not exist.

20.



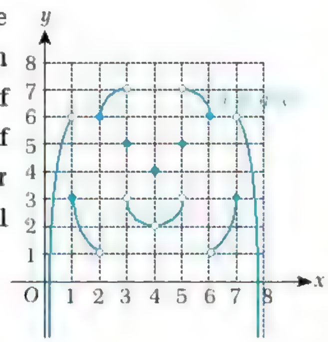
The graph of a function f: $(-3, 3) \to \mathbb{R}$ is shown in the figure. At which integer value(s) of the domain does a limit exist?

21. $f: \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} x-2 & \text{if} \quad x > 3 \\ 3 & \text{if} \quad x = 3 \\ -x+4 & \text{if} \quad 1 \le x < 3 \\ 1 & \text{if} \quad x < 1 \end{cases}$

is given. Calculate the limits.

a. $\lim\limits_{x\to 1} f(x)$ b. $\lim\limits_{x\to 3} f(x)$

22. The figure shows the graph of a function $g(x)$. Find the sum of the left-hand limits of $g(x)$ at the integer values in the interval $(0, 8)$.



E. Limits of Special Functions

23. Calculate the limits.

a. $\lim\limits_{x\to 2^+} \dfrac{|x-2|}{|2-x|}$ b. $\lim\limits_{x\to 3} |9-x^2|$

c. $\lim\limits_{x\to 0^+} \dfrac{|x|}{x^2+x}$ d. $\lim\limits_{x\to 5^-} \dfrac{|x-5|}{x-5}$

e. $\lim\limits_{x\to \frac{\pi}{2}^+} \dfrac{|\cos x|}{\cos x}$ f. $\lim\limits_{x\to 0^-} \dfrac{|x|}{x}$

24. Calculate the limits.

a. $\lim\limits_{x\to 2^-}(x^2 - \operatorname{sgn}(x-2))$ b. $\lim\limits_{x\to \frac{\pi}{2}^+}(\operatorname{sgn}(\cos x))$

c. $\lim\limits_{x\to 2^+} \dfrac{|4-x^2|}{\operatorname{sgn}|2-x|-x}$ d. $\lim\limits_{x\to 5^-} \dfrac{x^2-2x}{x-3\operatorname{sgn}(x-5)}$

e. $\lim\limits_{x\to 3} \dfrac{3x+2}{\operatorname{sgn}(-x^2+6x-9)}$ f. $\lim\limits_{x\to 1} \dfrac{1-x}{\operatorname{sgn}(x^2-3x+2)}$

25. Calculate the limits.

a. $\lim\limits_{x\to 2^-}([\![x]\!]+3\operatorname{sgn} x)$ b. $\lim\limits_{x\to 3}([\![x]\!]+x)$

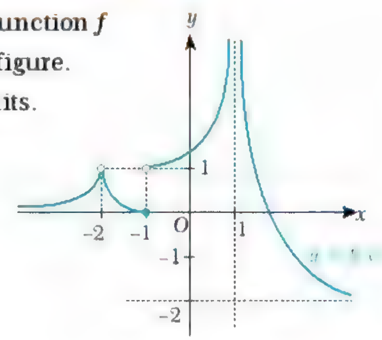
c. $\lim\limits_{x\to \pi^+} \dfrac{[\![\sin x]\!]}{\sin x+1}$ d. $\lim\limits_{x\to \pi^+}([\![\cos x]\!]+\cos|x|)$

e. $\lim\limits_{x\to \sqrt{3}} [\![x^2-3]\!]$ f. $\lim\limits_{x\to 3^+} \dfrac{3-[\![x]\!]}{x-3}$

g. $\lim\limits_{x\to 2^-} [\![x^5]\!]$

F. Limits Involving Infinity

26. The graph of a function f is shown in the figure. Evaluate the limits.



a. $\lim\limits_{x\to +\infty} f(x)$

b. $\lim\limits_{x\to -\infty} f(x)$

c. $\lim\limits_{x\to 1} f(x)$

d. $\lim\limits_{x\to -1} f(x)$

e. $\lim\limits_{x\to -2} f(x)$

27. Calculate the limits.

a. $\lim\limits_{x\to +\infty} \dfrac{2x^2+1}{3x^2+4}$ b. $\lim\limits_{x\to +\infty} (\dfrac{1}{x}+1)$

c. $\lim\limits_{x\to -\infty} (\dfrac{x^2-1}{x^2})$ d. $\lim\limits_{x\to -\infty} (\dfrac{2|x|}{x})$

e. $\lim\limits_{x\to -\infty} \dfrac{\sqrt{4x^2+x-5}}{\sqrt[3]{x^3+1}}$ f. $\lim\limits_{x\to +\infty} (1+\cos\dfrac{1}{x})$

28. Calculate the limits.

a. $\lim\limits_{x \to 2^+} \frac{1}{x^2 - 4}$ b. $\lim\limits_{x \to 0} \frac{1}{|x|}$

c. $\lim\limits_{x \to 3^-} \frac{x}{x^2 - 9}$ d. $\lim\limits_{x \to 3} \frac{x^3 + 2}{x|x-1| - 6}$

e. $\lim\limits_{x \to \frac{\pi}{2}^-} (x - \tan x)$

29. $f: \mathbb{R} \to \mathbb{R},\ f(x) = \frac{x+2}{x(x+1)(x-2)}$ is given.

Calculate the limits.

a. $\lim\limits_{x \to -\infty} f(x)$ b. $\lim\limits_{x \to -1^-} f(x)$

c. $\lim\limits_{x \to -1^+} f(x)$ d. $\lim\limits_{x \to 0^-} f(x)$

e. $\lim\limits_{x \to 0^+} f(x)$ f. $\lim\limits_{x \to 2^-} f(x)$

g. $\lim\limits_{x \to 2^+} f(x)$ h. $\lim\limits_{x \to +\infty} f(x)$

30. Calculate the limits.

a. $\lim\limits_{x \to +\infty} (-x^2 + 1)$ b. $\lim\limits_{x \to -\infty} \frac{x^3}{x^2 + 5}$

c. $\lim\limits_{x \to -\infty} \frac{2x^3 - x^2 + 1}{x^2 + 2x + 1}$ d. $\lim\limits_{x \to +\infty} \frac{1}{1 - 5^{\frac{1}{x}}}$

31. Evaluate $\lim\limits_{x \to 2} \frac{f(x) + \sqrt{f(x)}}{f(x) - 5}$ given $\lim\limits_{x \to 2} f(x) = 4$.

32. Use the limit combination theorem to calculate each limit.

a. $\lim\limits_{x \to 3} \frac{2x - 1}{x^2 + 1}$ b. $\lim\limits_{x \to 2} (4x^2 - 3x + 6)$

c. $\lim\limits_{x \to \pi} \sqrt[5]{\frac{x - \pi}{x + \pi}}$ d. $\lim\limits_{x \to 2} \sqrt{2x^2 + 4}$

e. $\lim\limits_{x \to 7} \frac{\sqrt[3]{x^2 - 1}}{x - 1}$

Mixed Problems

33. Calculate the limits.

a. $\lim\limits_{x \to 2^+} \left(\frac{[\![x]\!]^2 - 4}{x - 2} + x - 1\right)$ b. $\lim\limits_{x \to \frac{\pi}{2}^-} x^{[\![\cos x]\!]}$

c. $\lim\limits_{x \to 0^-} \frac{[\![x]\!]}{x - [\![x]\!]}$ d. $\lim\limits_{x \to 1^+} \frac{|1 - x|}{x^2 - 1} - [\![x - 1]\!]$

e. $\lim\limits_{x \to 3^+} \frac{3 - [\![x]\!]}{x - 3}$

f. $\lim\limits_{x \to 2^-} (|x - 2| + \operatorname{sgn}(x - 2) + [\![x + 2]\!])$

34. Calculate the limits.

a. $\lim\limits_{x \to 5^-} \left(25^{\frac{1}{x-5}}\right)$ b. $\lim\limits_{x \to 0^-} \frac{2 + 3^{\frac{1}{x}}}{1 + 4^{\frac{1}{x}}}$

35. Evaluate $\lim\limits_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$ if

$$f(x) = \begin{cases} \operatorname{sgn}(x - 2) & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2. \end{cases}$$

36. Evaluate $\lim\limits_{x \to 1} f(x)$ if

$$f(x) = \begin{cases} [\![x - 2]\!] - [\![x - 1]\!] & \text{if } x > 1 \\ \operatorname{sgn} \frac{x - 1}{|x - 1|} & \text{if } x < 1. \end{cases}$$

37. Calculate the limits.

a. $\lim\limits_{x \to +\infty} \operatorname{sgn}\left(\frac{3x + 5}{1 - x}\right)$ b. $\lim\limits_{x \to 3^-} \frac{[\![x^2]\!] - 9}{x^2 - 9}$

c. $\lim\limits_{x \to -\infty} \left(\operatorname{sgn}\left(\frac{x}{5 - x^2}\right)\right)$ d. $\lim\limits_{x \to 0^-} \frac{[\![1 - x]\!]}{|x|}$

2 INDETERMINATE FORMS

So far we have studied the concept and formal definition of the limit of a function. We have seen that even if a given function f is not defined at x_0, in some cases the limit of the function may exist as the point x approaches the point x_0 or infinity.

For example, the function $f(x) = \dfrac{x^2 - 9}{x - 3}$ is not defined at $x_0 = 3$: when $x_0 = 3$, we get the value $\dfrac{0}{0}$. But we know that $\lim\limits_{x \to 3} f(x)$ is not the same as $f(3)$. It doesn't matter what happens at $x_0 = 3$; the important thing is what happens as x approaches this point. So the limit of a function may exist even at undefined points.

Similarly, the function $f(x) = \dfrac{x^2 + x + 1}{2x^2 - 3}$ approaches the value $\dfrac{\infty}{\infty}$ as x approaches infinity.

$\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$ are two examples of indeterminate forms. In the following section we will look at how to evaluate limits which have the indeterminate forms $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, and 1^{∞}.

A. $\dfrac{0}{0}$ AS A LIMIT

Let f and g be two functions and let $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$ and $g(x_0) = 0$.

Then $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}$ has the indeterminate form $\dfrac{0}{0}$.

In this case there exists a function $h(x)$ which is a common factor of the functions f and g such that $f(x) = f_1(x) \cdot h(x)$, $g(x) = g_1(x) \cdot h(x)$ and $h(x_0) = 0$.

So we can write

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f_1(x) \cdot h(x)}{g_1(x) \cdot h(x)}.$$

Since $x \neq x_0$ (it is very close to, but not equal to, x_0) we can cancel the factors $h(x)$ and so

$$\lim_{x \to x_0} \frac{f_1(x) \cdot h(x)}{g_1(x) \cdot h(x)} = \lim_{x \to x_0} \frac{f_1(x)}{g_1(x)} = \frac{f_1(x_0)}{g_1(x_0)}$$

which is the result of the limit.

EXAMPLE 21 Find $\lim_{x\to 3} \frac{x^2-9}{x-3}$.

Solution As x approaches 3, the quotient approaches the indeterminate form $\frac{0}{0}$. However, by factoring the expression in the numerator we get

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x\to 3}(x+3) = 6.$$

Remember that we can cancel the factor $\frac{(x-3)}{(x-3)}$ because $x \neq 3$ at the limit (it is only approaching this point).

EXAMPLE 22 Find $\lim_{x\to 2} \frac{x^2-5x+6}{x^2-x-2}$.

Solution This limit has the indeterminate form $\frac{0}{0}$ as x approaches 2, so we need to factor the numerator and denominator and find the common factor:

$$\lim_{x\to 2} \frac{x^2-5x+6}{x^2-x-2} = \lim_{x\to 2} \frac{(x-2)(x-3)}{(x-2)(x+1)} = \lim_{x\to 2} \frac{(x-3)}{(x+1)} = -\frac{1}{3}.$$

EXAMPLE 23 Find $\lim_{x\to 1} \frac{\sqrt{x}-1}{x^3-1}$.

Solution The limit has the indeterminate form $\frac{0}{0}$ as x approaches 1.

We can begin by multiplying both the numerator and the denominator by the conjugate of $\sqrt{x}-1$:

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x^3-1} = \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x^3-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{(x-1)}{(x^3-1)(\sqrt{x}+1)}.$$

We know that $x^3-1 = (x-1)(x^2+x+1)$, so

$$\lim_{x\to 1} \frac{(x-1)}{(x^3-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{(x-1)}{(x-1)(x^2+x+1)(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{1}{(x^2+x+1)(\sqrt{x}+1)} = \frac{1}{6}.$$

EXAMPLE 24 Find $\lim\limits_{x\to\infty} \dfrac{\frac{2}{x}}{\left(2+\frac{2}{x}\right)^2-4}$.

Solution The indeterminate form is $\frac{0}{0}$. Let us factor the denominator:

$$\lim_{x\to\infty} \frac{\frac{2}{x}}{\left(2+\frac{2}{x}\right)^2-4} = \lim_{x\to\infty} \frac{\frac{2}{x}}{\left(2+\frac{2}{x}-2\right)\left(2+\frac{2}{x}+2\right)} = \lim_{x\to\infty} \frac{\left(\frac{2}{x}\right)}{\left(\frac{2}{x}\right)\left(4+\frac{2}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{\left(4+\frac{2}{x}\right)} = \frac{1}{4}.$$

Check Yourself 9

Calculate the limits.

1. $\lim\limits_{x\to 4} \dfrac{x^2-2x-8}{x^2-3x-4}$
2. $\lim\limits_{x\to 3} \dfrac{x^2-9}{x^3-27}$
3. $\lim\limits_{x\to 1} \dfrac{x^3-x^2-x+1}{3x^2-6x+3}$
4. $\lim\limits_{x\to 2} \dfrac{\sqrt{x+2}-2}{x-2}$
5. $\lim\limits_{x\to 16} \dfrac{\sqrt{x}-4}{\sqrt[4]{x}-2}$
6. $\lim\limits_{x\to 0} \dfrac{\sqrt{1-x^4}-\sqrt{1+x^4}}{x^4}$
7. $\lim\limits_{x\to 0} \dfrac{\sqrt{4+3x}-2}{x}$
8. $\lim\limits_{x\to 2} \dfrac{4-2^x}{4^x-16}$

Answers

1. $\frac{6}{5}$ 2. $\frac{2}{9}$ 3. $\frac{2}{3}$ 4. $\frac{1}{4}$ 5. 4 6. -1 7. $\frac{3}{4}$ 8. $-\frac{1}{8}$

1. The Indeterminate Form $\frac{0}{0}$ in Trigonometric Functions

As x approaches a point x_0, the limit of a trigonometric function is the image of x_0.

For example, $\lim\limits_{x\to 0} \cos x = \cos 0 = 1$, $\lim\limits_{x\to 0} \sin x = \sin 0 = 0$, and $\lim\limits_{x\to \frac{\pi}{4}} \tan x = \tan\frac{\pi}{4} = 1$.

But what about a limit such as $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$?

This limit has the indeterminate form $\frac{0}{0}$, and we need to use a different approach to evaluate it. Look at the following figure.

In the figure, $PR = \sin x$, $OR = \cos x$, and $QA = \tan x$.

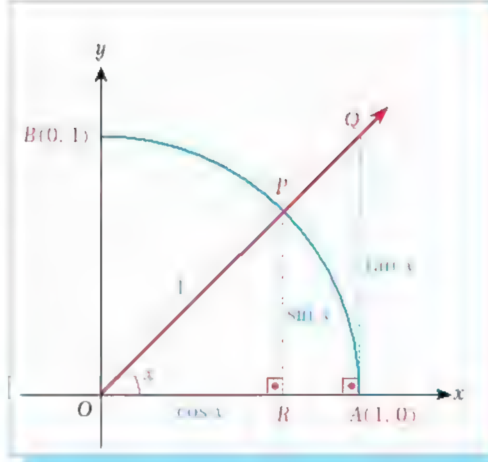
Now let us consider the areas of ΔOPR and ΔOQA and the area of the sector OPA.

We can see that $A(\Delta OPR) < A(OPA) < A(\Delta OQA)$.

By using the formulas for the area of a triangle and a circle we get

$$\frac{1}{2}\cos x \cdot \sin x < \pi \cdot 1^2 \cdot \frac{x}{2\pi} < \frac{1}{2} \cdot 1 \cdot \tan x, \text{ i.e.}$$

$$\cos x \cdot \sin x < x < \tan x. \quad (1)$$



For $x \to 0^+$, $\sin x > 0$ and when we divide each side of inequality (1) by $\sin x$ we get

$$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}. \quad (2)$$

Now let us take the limit of each side of (2) as $x \to 0^+$:

$$\lim_{x\to 0^+} \cos x \le \lim_{x\to 0^+} \frac{x}{\sin x} \le \lim_{x\to 0^+} \frac{1}{\cos x}.$$

We can rewrite this as $1 \le \lim\limits_{x\to 0^+} \frac{x}{\sin x} \le 1$, and so we can say that $\lim\limits_{x\to 0^+} \frac{x}{\sin x} = 1$.

Similarly, $\lim\limits_{x\to 0^-} \cos x \le \lim\limits_{x\to 0^-} \frac{x}{\sin x} \le \lim\limits_{x\to 0^-} \frac{1}{\cos x}$, i.e. $1 \le \lim\limits_{x\to 0^-} \frac{x}{\sin x} \le 1$ and so $\lim\limits_{x\to 0^-} \frac{x}{\sin x} = 1$.

In conclusion, $\lim\limits_{x\to 0} \frac{x}{\sin x} = 1$.

What about $\lim\limits_{x\to 0} \frac{\sin x}{x}$?

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{1}{\frac{x}{\sin x}} = \frac{1}{\lim\limits_{x\to 0} \frac{x}{\sin x}} = \frac{1}{1} = 1$$

When we are calculating the limit of a trigonometric function which involves the indeterminate form $\frac{0}{0}$, we can use the rule $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

Conclusion

1. $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$

2. $\lim\limits_{x\to 0} \frac{\sin ax}{bx} = \frac{a}{b}$ and $\lim\limits_{x\to 0} \frac{\tan ax}{bx} = \frac{a}{b}$

3. $\lim\limits_{x \to 0} \dfrac{\sin ax}{\sin bx} = \dfrac{a}{b}$ and $\lim\limits_{x \to 0} \dfrac{\tan ax}{\tan bx} = \dfrac{a}{b}$

4. $\lim\limits_{x \to 0} \dfrac{\sin ax}{\tan bx} = \dfrac{a}{b}$ and $\lim\limits_{x \to 0} \dfrac{\tan ax}{\sin bx} = \dfrac{a}{b}$.

Proof

1. $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x \to 0} \dfrac{1}{\cos x} = 1 \cdot 1 = 1.$

2. As $x \to 0$, let $ax = u$ so $x = \dfrac{u}{a}$ and consider $u \to 0$.

$$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{1}{b} \cdot \lim_{x \to 0} \frac{\sin ax}{x} = \frac{1}{b} \cdot \lim_{u \to 0} \frac{\sin u}{\frac{u}{a}} = \frac{a}{b} \cdot \lim_{u \to 0} \frac{\sin u}{u} = \frac{a}{b} \cdot 1 = \frac{a}{b}$$

3. $$\lim_{x \to 0} \frac{\tan ax}{\tan bx} = \lim_{x \to 0} \frac{\frac{\tan ax}{ax} \cdot ax}{\frac{\tan bx}{bx} \cdot bx} = \lim_{x \to 0} \frac{\frac{\tan ax}{ax}}{\frac{\tan bx}{bx}} \cdot \frac{a}{b} = \frac{1}{1} \cdot \frac{a}{b} = \frac{a}{b}$$

4. $$\lim_{x \to 0} \frac{\sin ax}{\tan bx} = \lim_{x \to 0} \frac{ax \cdot \frac{\sin ax}{ax}}{bx \cdot \frac{\tan bx}{bx}} = \frac{a}{b} \cdot \lim_{x \to 0} \frac{\frac{\sin ax}{ax}}{\frac{\tan bx}{bx}} = \frac{a}{b} \cdot \frac{\lim\limits_{x \to 0} \frac{\sin ax}{ax}}{\lim\limits_{x \to 0} \frac{\tan bx}{bx}} = \frac{a}{b} \cdot \frac{1}{1} = \frac{a}{b}$$

EXAMPLE 25 Find $\lim\limits_{x \to 2} \dfrac{4 - x^2}{\sin(2 - x)}$.

Solution $\lim\limits_{x \to 2} \dfrac{4 - x^2}{\sin(2 - x)} = \lim\limits_{x \to 2} \dfrac{(2 - x)(2 + x)}{\sin(2 - x)} = \lim\limits_{x \to 2} \dfrac{(2 - x)}{\sin(2 - x)} \cdot \lim\limits_{x \to 2}(2 + x) = 1 \cdot 4 = 4$

EXAMPLE 26 Find $\lim\limits_{x \to \pi} \dfrac{\tan x}{3(x - \pi)}$.

Solution Since $\tan x = -\tan(\pi - x)$,

$$\lim_{x \to \pi} \frac{\tan x}{3(x - \pi)} = \lim_{x \to \pi} \frac{-\tan(\pi - x)}{-3(\pi - x)} = \frac{1}{3} \lim_{x \to \pi} \frac{\tan(\pi - x)}{(\pi - x)} = \frac{1}{3} \cdot 1 = \frac{1}{3}.$$

EXAMPLE 27 Find $\lim\limits_{x\to 0}\dfrac{\sin^2\frac{x}{2}}{3x^2}$.

Solution $$\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{3x^2}=\frac{1}{3}\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\frac{4x^2}{4}}=\frac{1}{3}\cdot\frac{1}{4}\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}=\frac{1}{12}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{12}\cdot 1^2=\frac{1}{12}$$

EXAMPLE 28 Evaluate $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{8x^2}$.

Solution Since $\cos 2x = 1-2\sin^2 x$,

$$\lim_{x\to 0}\frac{1-\cos 2x}{8x^2}=\lim_{x\to 0}\frac{1-(1-2\sin^2 x)}{8x^2}=\lim_{x\to 0}\frac{1-1+2\sin^2 x}{8x^2}=\lim_{x\to 0}\frac{\sin^2 x}{4x^2}=\frac{1}{4}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$

$$=\frac{1}{4}\cdot 1^2=4.$$

EXAMPLE 29 Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos x}{\pi-2x}$.

Solution Since $\cos x=\sin(\frac{\pi}{2}-x)$ we can write

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{(\pi-2x)}=\lim_{x\to\frac{\pi}{2}}\frac{\sin(\frac{\pi}{2}-x)}{(\pi-2x)}=\lim_{x\to\frac{\pi}{2}}\frac{\sin\frac{1}{2}(\pi-2x)}{(\pi-2x)}=\frac{1}{2}.$$

Check Yourself 10

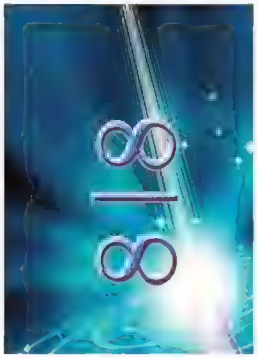
Calculate the limits.

1. $\lim\limits_{x\to 0}\dfrac{\tan 2x}{\tan 5x}$
2. $\lim\limits_{x\to 0}\dfrac{\sin x}{\tan 3x}$
3. $\lim\limits_{x\to 0}\dfrac{\sin\frac{x}{2}}{3x}$
4. $\lim\limits_{x\to 1}\dfrac{\sin(2x-2)}{4x-4}$
5. $\lim\limits_{x\to 0}\dfrac{\sin^2 2x}{x^2}$
6. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x\cdot\cos 5x}$
7. $\lim\limits_{x\to 1}\dfrac{\tan\pi x}{1-x}$
8. $\lim\limits_{x\to 0}\dfrac{3x^2}{1-\cos 2x}$
9. $\lim\limits_{x\to 0}\dfrac{\cos(3x+\frac{3\pi}{2})}{\sin(\pi-2x)}$

Answers

1. $\frac{2}{5}$ 2. $\frac{1}{3}$ 3. $\frac{1}{6}$ 4. $\frac{1}{2}$ 5. 4 6. 1 7. $-\pi$ 8. $\frac{3}{2}$ 9. $\frac{3}{2}$

B. $\frac{\infty}{\infty}$ AS A LIMIT



Let f and g be two functions and let $x_0 \in \mathbb{R}$ such that as x approaches x_0, both $f(x)$ and $g(x)$ approach positive or negative infinity, i.e. $\lim\limits_{x\to x_0} f(x) = \pm\infty$ and $\lim\limits_{x\to x_0} g(x) = \pm\infty$.

Then $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$ has the indeterminate form $\dfrac{\infty}{\infty}$. We can find the limit of such functions as follows:

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ and

$Q(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0$ be two polynomial functions, then

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_o}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_o}$$

$$= \lim_{x\to\pm\infty} \frac{x^n\left(a_n + \dfrac{a_{n-1}}{x} + \ldots + \dfrac{a_o}{x^n}\right)}{x^m\left(b_m + \dfrac{b_{m-1}}{x} + \ldots + \dfrac{b_o}{x^m}\right)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}.$$

As x approaches $\pm\infty$, the limit of each rational expression approaches zero, so we have eliminated the rational expressions. In conclusion, we can write

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \begin{cases} \dfrac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \\ \pm\infty & \text{if } n > m. \end{cases}$$

The degree of a polynomial is the highest power of the variable in the polynomial. The leading coefficient is the coefficient of the term with the highest degree. For example, $x - 3x^2$ has degree 2 and its leading coefficient is -3.

This is the same as saying the following:

1. If the degrees of the polynomials are equal then the limit is the ratio of the leading coefficients.
2. If the degree of the polynomial in the denominator is greater than the degree of the polynomial in the numerator then the limit is zero.
3. If the degree of the polynomial in the denominator is less than the degree of the polynomial in the numerator then the limit is infinite (i.e. it approaches positive or negative infinity).

EXAMPLE 30 Evaluate the limits.

a. $\lim\limits_{x\to\infty}\dfrac{5x^3-2x^2+x-1}{4x^3-x^2+12}$ b. $\lim\limits_{x\to\infty}\dfrac{-2x^3-1}{x^5+x^3+1}$ c. $\lim\limits_{x\to-\infty}\dfrac{x^2+2}{x+4}$

Solution The indeterminate form is $\dfrac{\infty}{\infty}$.

a. $\lim\limits_{x\to\infty}\dfrac{5x^3-2x^2+x-1}{4x^3-x^2+12}=\dfrac{5}{4}$ because the polynomials have the same degree. Alternatively, we can calculate

$$\lim_{x\to\infty}\frac{5x^3-2x^2+x-1}{4x^3-x^2+12}=\lim_{x\to\infty}\frac{x^3(5-\frac{2}{x}+\frac{1}{x^2}-\frac{1}{x^3})}{x^3(4-\frac{1}{x}+\frac{12}{x^3})}=\lim_{x\to\infty}\frac{5x^3}{4x^3}=\frac{5}{4}.$$

b. $\lim\limits_{x\to\infty}\dfrac{-2x^3-1}{x^5+x^3+1}=0$ because the degree of the polynomial in the denominator is bigger. We can also calculate this limit:

$$\lim_{x\to\infty}\frac{-2x^3-1}{x^5+x^3+1}=\lim_{x\to\infty}\frac{x^3(-2-\frac{1}{x^3})}{x^5(1+\frac{1}{x^2}+\frac{1}{x^5})}=\lim_{x\to\infty}\frac{-2x^3}{x^5}=\lim_{x\to\infty}\frac{-2}{x^2}=0.$$

c. $\lim\limits_{x\to-\infty}\dfrac{x^2+2}{x+4}=-\infty$ because the degree of the polynomial in the numerator is bigger. We can also calculate this directly:

$$\lim_{x\to-\infty}\frac{x^2+2}{x+4}=\lim_{x\to-\infty}\frac{x^2\left(1+\frac{2}{x^2}\right)}{x\left(1+\frac{4}{x}\right)}=\lim_{x\to-\infty}\frac{x^2}{x}=\lim_{x\to-\infty}x=-\infty.$$

EXAMPLE 31 Calculate the limits.

a. $\lim\limits_{x\to\infty}\dfrac{2x+4}{\sqrt{9x^2+2}}$ b. $\lim\limits_{x\to-\infty}\dfrac{x-\sqrt{4x^2-1}}{\sqrt{x^2+2x+2}}$

Solution a. The indeterminate form is $\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{2x+4}{\sqrt{9x^2+2}}=\lim_{x\to\infty}\frac{x\left(2+\frac{4}{x}\right)}{\sqrt{9x^2\left(1+\frac{2}{9x^2}\right)}}=\lim_{x\to\infty}\frac{x\left(2+\frac{4}{x}\right)}{3|x|\sqrt{1+\frac{2}{9x^2}}}=\lim_{x\to\infty}\frac{x\left(2+\frac{4}{x}\right)}{3x\sqrt{1+\frac{2}{9x^2}}}=\frac{2}{3}$$

b. The indeterminate form is $\frac{\infty}{\infty}$.

$$\lim_{x\to-\infty}\frac{x-\sqrt{4x^2-1}}{\sqrt{x^2+2x+2}}=\lim_{x\to-\infty}\frac{x-|x|\sqrt{\left(4-\frac{1}{x^2}\right)}}{|x|\sqrt{1+\frac{2}{x}+\frac{2}{x^2}}}=\lim_{x\to-\infty}\frac{x-(-x)\sqrt{\left(4-\frac{1}{x^2}\right)}}{-x\sqrt{1+\frac{2}{x}+\frac{2}{x^2}}}$$

$$=\lim_{x\to-\infty}\frac{x+x\sqrt{4-\frac{1}{x^2}}}{-x\sqrt{1+\frac{2}{x}+\frac{2}{x^2}}}=\lim_{x\to-\infty}\frac{x\left(1+\sqrt{4-\frac{1}{x^2}}\right)}{-x\sqrt{1+\frac{2}{x}+\frac{2}{x^2}}}=\frac{1+\sqrt{4}}{-\sqrt{1}}=-3$$

EXAMPLE 32 Find $\lim_{x\to 0}\frac{\cot 2x}{\cot 3x}$.

Solution The indeterminate form is $\frac{\infty}{\infty}$. We can turn it into the indeterminate form $\frac{0}{0}$. $\lim_{x\to 0}\frac{\cot 2x}{\cot 3x}=\lim_{x\to 0}\frac{\frac{1}{\tan 2x}}{\frac{1}{\tan 3x}}=\lim_{x\to 0}\frac{\tan 3x}{\tan 2x}=\frac{3}{2}$

Sometimes we calculate the indeterminate form $\frac{\infty}{\infty}$ by transforming it into the indeterminate form $\frac{0}{0}$.

EXAMPLE 33 Find $\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\cot(x-\frac{\pi}{2})}$.

Solution As x approaches $\frac{\pi}{2}$, the limit approaches the indeterminate form $\frac{\infty}{\infty}$. We know that $\cot(x-\frac{\pi}{2})=-\tan x$ and so $\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\cot\left(x-\frac{\pi}{2}\right)}=\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{-\tan x}=-\lim_{x\to\frac{\pi}{2}}\frac{\tan x}{\tan x}=-1$.

EXAMPLE 34 Find $\lim_{x\to\infty}(\frac{2x}{\cot\frac{1}{x}})$.

Solution The indeterminate form is $\frac{\infty}{\infty}$. We can write $\cot\frac{1}{x}$ as $\frac{1}{\tan\frac{1}{x}}$.

Then $\lim_{x\to\infty}(\frac{2x}{\cot\frac{1}{x}})=2\lim_{x\to\infty}(\frac{x}{\frac{1}{\tan\frac{1}{x}}})=2\lim_{x\to\infty}(\frac{\tan\frac{1}{x}}{\frac{1}{x}})=2\cdot 1=2.$

EXAMPLE 35 Find $\lim\limits_{x\to\infty}\dfrac{2^x+5^{x+1}}{3^x+5^x}$.

Solution The indeterminate form is $\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty}\frac{2^x+5^{x+1}}{3^x+5^x}=\lim_{x\to\infty}\frac{5^x\left((\frac{2}{5})^x+5\right)}{5^x\left((\frac{3}{5})^x+1\right)}=5,\ \text{since}\ \lim_{x\to\infty}(\frac{2}{5})^x=0\ \text{and}\ \lim_{x\to\infty}(\frac{3}{5})^x=0\ .$$

Check Yourself 11

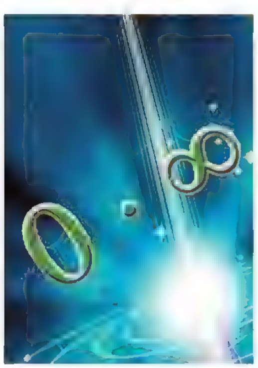
Calculate the limits.

1. $\lim\limits_{x\to\infty}\dfrac{1-x^2+7x^3}{5+x-x^3}$
2. $\lim\limits_{x\to-\infty}\dfrac{x^5+2x+7}{x^2+12x-1}$
3. $\lim\limits_{x\to-\infty}\dfrac{x^2+2x-12}{1-x^3}$
4. $\lim\limits_{x\to-\infty}\dfrac{3x^2+1}{\sqrt{4x^2+4}}$
5. $\lim\limits_{x\to\infty}\dfrac{x-\sqrt{9x^2-1}}{\sqrt{x^2+1}+3x}$
6. $\lim\limits_{x\to 0}\dfrac{\cot 5x}{\cot 6x}$
7. $\lim\limits_{x\to\pi}\dfrac{\tan\frac{x}{2}}{\cot(x-\pi)}$
8. $\lim\limits_{x\to\infty}\dfrac{x^2+\cos x}{3x^2}$
9. $\lim\limits_{x\to\infty}\dfrac{7^{x+2}+3^x}{5^x+7^x}$

Answers

1. -7 2. $-\infty$ 3. 0 4. ∞ 5. $-\dfrac{1}{2}$ 6. $\dfrac{6}{5}$ 7. -2 8. $\dfrac{1}{3}$ 9. 49

C. $0\cdot\infty$ AS A LIMIT



Let f and g be two functions and let $x_0\in\mathbb{R}$, and let us assume that as x approaches x_0, $f(x)$ approaches zero but $g(x)$ approaches positive or negative infinity, i.e.

$\lim\limits_{x\to x_0}f(x)=0$ and $\lim\limits_{x\to x_0}g(x)=\pm\infty$

Then $\lim\limits_{x\to x_0}[f(x)\cdot g(x)$ has the indeterminate form $0\cdot\infty$.

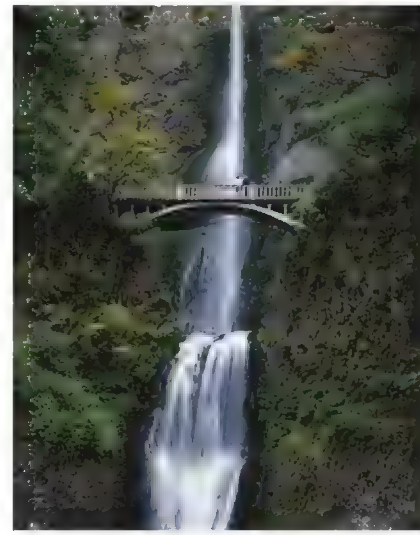
We can find the limit of such functions by transforming the indeterminate form $0\cdot\infty$ into the indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ as follows:

$\lim\limits_{x\to x_0}\left[f(x)\cdot g(x)\right]=\lim\limits_{x\to x_0}\dfrac{f(x)}{\dfrac{1}{g(x)}}$ has the indeterminate form $\dfrac{0}{0}$.

$\lim\limits_{x\to x_0}\left[f(x)\cdot g(x)\right]=\lim\limits_{x\to x_0}\dfrac{g(x)}{\dfrac{1}{f(x)}}$ has the indeterminate form $\dfrac{\infty}{\infty}$.

EXAMPLE 36 Find $\lim_{x\to\infty}(x\cdot\sin\frac{1}{x})$.

Solution The function has the indeterminate form $\infty\cdot 0$. We can transform it into the form $\frac{0}{0}$ by writing



$$\lim_{x\to\infty}(x\cdot\sin\frac{1}{x})=\lim_{x\to\infty}(\frac{\sin\frac{1}{x}}{\frac{1}{x}}).$$

Let us take $\frac{1}{x}=t$. As $x\to\infty$, $\frac{1}{x}\to 0$ so $t\to 0$,

and so $\lim_{x\to\infty}(\frac{\sin\frac{1}{x}}{\frac{1}{x}})=\lim_{t\to 0}(\frac{\sin t}{t})=1$.

EXAMPLE 37 Find $\lim_{x\to 0}(2x\cdot\cot x)$.

Solution The indeterminate form is $0\cdot\infty$.

$$\lim_{x\to 0}(2x\cdot\cot x)=\lim_{x\to 0}2x\cdot\frac{1}{\tan x}=2\cdot\lim_{x\to 0}\frac{x}{\tan x}=2\cdot 1=2.$$

EXAMPLE 38 Find $\lim_{x\to-\infty}\frac{1}{x+5}\cdot(3x-2)$.

Solution The indeterminate form is $0\cdot\infty$.

$$\lim_{x\to-\infty}\frac{1}{x+5}\cdot(3x-2)=\lim_{x\to-\infty}\frac{3x-2}{x+5}=3.$$

EXAMPLE 39 Find $\lim_{x\to\frac{\pi}{2}}(\pi-2x)\cdot\tan x$.

Solution The indeterminate form is $0\cdot\infty$.

$$\lim_{x\to\frac{\pi}{2}}(\pi-2x)\cdot\tan x=\lim_{x\to\frac{\pi}{2}}(\pi-2x)\cdot\cot(\frac{\pi}{2}-x)=\lim_{x\to\frac{\pi}{2}}\frac{(\pi-2x)}{\tan(\frac{\pi}{2}-x)}=\lim_{x\to\frac{\pi}{2}}\frac{2(\frac{\pi}{2}-x)}{\tan(\frac{\pi}{2}-x)}=2.$$

EXAMPLE 40 Find $\lim_{x\to\infty} x^2(\cos\frac{1}{x}-1)$.

Solution The indeterminate form is $\infty\cdot 0$. We know that $\cos 2x = 1-2\sin^2 x$, and so

$$\lim_{x\to\infty} x^2\left(\cos\frac{1}{x}-1\right)=\lim_{x\to\infty} x^2\left(1-2\sin^2\left(\frac{1}{2x}\right)-1\right)=-2\lim_{x\to\infty} x^2\sin^2\left(\frac{1}{2x}\right)=-2\lim_{x\to\infty}\frac{\sin^2\left(\frac{1}{2x}\right)}{\frac{1}{x^2}}$$

$$=-2\lim_{x\to\infty}\left(\frac{\sin\left(\frac{1}{2x}\right)}{\frac{1}{x}}\right)^2=-2\left(\frac{1}{2}\right)^2=-\frac{1}{2}.$$

Check Yourself 12

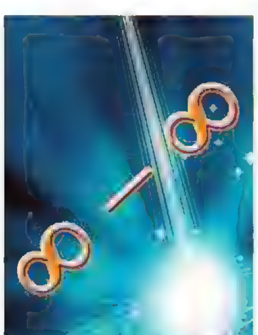
Calculate the limits.

1. $\lim_{x\to\infty}(\frac{x}{3}\cdot\sin\frac{9}{x})$
2. $\lim_{x\to\infty}(x\cdot\tan\frac{2\pi}{x})$
3. $\lim_{x\to\infty}(x+2)\frac{1}{(x^2-1)}$
4. $\lim_{x\to\infty}\sqrt{x^2+5x+1}\cdot\frac{1}{x+4}$
5. $\lim_{x\to 0}(2x^2\cdot\cot x^2)$
6. $\lim_{x\to 0}(\frac{1}{x^2}\cdot\sin 2x\cdot\tan x)$
7. $\lim_{x\to\pi}((\pi-x)\cot 2x)$

Answers

1\. 3 2. 2π 3. 0 4. 1 5. 2 6. 2 7. $-\frac{1}{2}$

D. $\infty-\infty$ AS A LIMIT



Let f and g be two functions. Let $x_0\in\mathbb{R}$, and let us assume that as x approaches x_0, both $f(x)$ and $g(x)$ approach infinity, i.e.

$$\lim_{x\to x_0} f(x)=\infty \text{ and } \lim_{x\to x_0} g(x)=\infty.$$

Then $\lim_{x\to x_0}[f(x)-g(x)]$ has the indeterminate form $\infty-\infty$. We can find the limit of such functions by transforming the indeterminate form $\infty-\infty$ into the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, as shown in the following examples.

EXAMPLE 41 Find $\lim\limits_{x\to 1}\left(\dfrac{2}{x^2-1}-\dfrac{1}{x-1}\right)$.

Solution The indeterminate form is $\infty-\infty$ as x approaches 1. We can rewrite the function to transform the indeterminate form $\infty-\infty$ into the indeterminate form $\dfrac{0}{0}$:

$$\lim_{x\to 1}\left(\frac{2}{x^2-1}-\frac{1}{x-1}\right)=\lim_{x\to 1}\left(\frac{2}{(x-1)(x+1)}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{2-x-1}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1-x}{(x-1)(x+1)}.$$

Now we have the indeterminate form $\dfrac{0}{0}$ and we can calculate the limit:

$$\lim_{x\to 1}\frac{1-x}{(x-1)(x+1)}=\lim_{x\to 1}\frac{-1}{(x+1)}=-\frac{1}{2}.$$

EXAMPLE 42 Find $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{\tan x}\right)$.

Solution Both $\lim\limits_{x\to 0}\dfrac{1}{\sin x}$ and $\lim\limits_{x\to 0}\dfrac{1}{\tan x}$ have the limit ∞, so the indeterminate form is $\infty-\infty$.

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{\tan x}\right)=\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)=\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{\sin x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 x}{\sin x(1+\cos x)}=\lim_{x\to 0}\frac{\sin^2 x}{\sin x(1+\cos x)}=\lim_{x\to 0}\frac{\sin x}{1+\cos x}=\frac{0}{2}=0$$

EXAMPLE 43 Find $\lim\limits_{x\to\infty}(\sqrt{4x^2-6x+3}-\sqrt{4x^2+3x+1})$.

Solution The indeterminate form is $\infty-\infty$. Let us multiply and divide the expression by its conjugate. Then we get

$$\lim_{x\to\infty}\frac{(\sqrt{4x^2-6x+3}-\sqrt{4x^2+3x+1})(\sqrt{4x^2-6x+3}+\sqrt{4x^2+3x+1})}{(\sqrt{4x^2-6x+3}+\sqrt{4x^2+3x+1})}$$

$$=\lim_{x\to\infty}\frac{4x^2-6x+3-(4x^2+3x+1)}{\left(\sqrt{4x^2\left(1-\frac{6}{4x}+\frac{3}{4x^2}\right)}+\sqrt{4x^2\left(1+\frac{3}{4x}+\frac{1}{4x^2}\right)}\right)}=\lim_{x\to\infty}\frac{-9x+2}{|2x|+|2x|}$$

$$=\lim_{x\to\infty}\frac{-9x\left(1-\frac{2}{9x}\right)}{2x+2x}=\lim_{x\to\infty}\frac{-9x}{4x}=-\frac{9}{4}.$$

Theorem

Let $a>0$ be a real number and let f be a function of the form $f(x)=\sqrt{ax^2+bx+c}$. Then

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\sqrt{ax^2+bx+c}=\lim_{x\to\pm\infty}\sqrt{a}\left|x+\frac{b}{2a}\right|.$$

Proof

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\sqrt{ax^2+bx+c}=\lim_{x\to\pm\infty}\sqrt{a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)}=\lim_{x\to\pm\infty}\sqrt{a}\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}\quad (a>0)$$

$$=\lim_{x\to\pm\infty}\sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}$$

As $x\to\pm\infty$, the value of the constant expression $\frac{4ac-b^2}{4a^2}$ becomes negligible compared to the rest of the function. So we can omit the expression $\frac{4ac-b^2}{4a^2}$ and we have

$$\lim_{x\to\pm\infty}\sqrt{a}\sqrt{\left(x+\frac{b}{2a}\right)^2}=\lim_{x\to\pm\infty}\sqrt{a}\left|x+\frac{b}{2a}\right|.$$

EXAMPLE 44 Find $\lim_{x\to\infty}(\sqrt{4x^2+8x+1}-\sqrt{4x^2-3x-1})$.

Solution The indeterminate form is $\infty-\infty$.

We can use the theorem to calculate the limit:

$$\lim_{x\to\infty}(\sqrt{4x^2+8x+1}-\sqrt{4x^2-3x-1})=\lim_{x\to\infty}\left(\sqrt{4}\left|x+\frac{8}{2\cdot 4}\right|-\sqrt{4}\left|x+\frac{-3}{2\cdot 4}\right|\right)$$

$$=\lim_{x\to\infty}\left(2(x+1)-2\left(x-\frac{3}{8}\right)\right)=\lim_{x\to\infty}\left(2x+2-2x+\frac{6}{8}\right)$$

$$=2+\frac{6}{8}=\frac{22}{8}=\frac{11}{4}.$$

EXAMPLE 45 Find $\lim_{x\to-\infty}(\sqrt{x^2-2x-3}+x-4)$.

Solution The indeterminate form is $\infty-\infty$.

Use the theorem:

$$\lim_{x\to-\infty}(\sqrt{x^2-2x-3}+x-4)=\lim_{x\to-\infty}\left(\sqrt{1}\cdot\left|x+\frac{-2}{2\cdot 1}\right|+x-4\right)=\lim_{x\to-\infty}|x-1|+x-4$$

$$=\lim_{x\to-\infty}(-x+1+x-4)=-3.$$

EXAMPLE 46 Find $\lim\limits_{x\to\infty}\dfrac{\sqrt{4x^2+1}-\sqrt{x^2}}{\sqrt{9x^2+2}-\sqrt{x^2+2}}$.

Solution The indeterminate form is $\dfrac{\infty-\infty}{\infty-\infty}$. Let us use the theorem:

$$\lim_{x\to\infty}\frac{\sqrt{4x^2+1}-\sqrt{x^2}}{\sqrt{9x^2+2}-\sqrt{x^2+2}}=\lim_{x\to\infty}\frac{\sqrt{4}\left|x+\dfrac{0}{2\cdot 4}\right|-\sqrt{1}\left|x+\dfrac{0}{2\cdot 1}\right|}{\sqrt{9}\left|x+\dfrac{0}{2\cdot 9}\right|-\sqrt{1}\left|x+\dfrac{0}{2\cdot 1}\right|}=\lim_{x\to\infty}\frac{2|x|-|x|}{3|x|-|x|}=\lim_{x\to\infty}\frac{2x-x}{3x-x}$$

$$=\lim_{x\to\infty}\frac{x}{2x}=\frac{1}{2}.$$

Check Yourself 13

Calculate the limits.

1. $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}+\dfrac{2}{1-x^2}\right)$
2. $\lim\limits_{x\to\infty}\left(\dfrac{x^2}{2x-5}-\dfrac{x}{2}\right)$
3. $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{1}{\cos x}-\dfrac{1}{\cot x}\right)$
4. $\lim\limits_{x\to\frac{\pi}{2}}(\tan x-\sec x)$
5. $\lim\limits_{x\to\infty}(\sqrt{x^2+3x+1}-x)$
6. $\lim\limits_{x\to\infty}(\sqrt{x^2+4x+1}-\sqrt{x^2+3x+3})$
7. $\lim\limits_{x\to\infty}(\sqrt{x+2}-\sqrt{x-2})$
8. $\lim\limits_{x\to-\infty}(2x^2-\sqrt{x^2+1})$
9. $\lim\limits_{x\to-\infty}(\sqrt{x^2-8x+1}+x+4)$
10. $\lim\limits_{x\to\infty}(\log_3(9x+1)-\log_3(x+4))$
11. $\lim\limits_{x\to-\infty}(-2x-1-\sqrt{4x^2+3x+12})$

Answers

1. $\dfrac{1}{2}$ 2. $\dfrac{5}{4}$ 3. 0 4. 0 5. $\dfrac{3}{2}$ 6. $\dfrac{1}{2}$ 7. 0 8. ∞ 9. 8 10. 2 11. $-\dfrac{1}{4}$

E. 1^{∞} AS A LIMIT

Let f and g be two functions and let $x_0\in\mathbb{R}$, and let us assume that as x approaches x_0, $f(x)$ approaches 0 and $g(x)$ approaches infinity, i.e.

$$\lim_{x\to x_0}f(x)=0 \text{ and } \lim_{x\to x_0}g(x)=\infty.$$

Since $\lim\limits_{x\to x_0}(1+f(x))$ approaches 1 but is not equal to 1, the limit $\lim\limits_{x\to x_0}(1+f(x))^{g(x)}$ has the indeterminate form 1^{∞}. We can remove the indeterminate form 1^{∞} by using the following rule:

Let $\lim\limits_{x\to x_0}f(x)=0$, $\lim\limits_{x\to x_0}g(x)=\infty$ and $\lim\limits_{x\to x_0}f(x)\cdot g(x)=k$ $(k\in\mathbb{R})$. Then

$$\lim_{x\to x_0}(1+f(x))^{g(x)}=e^{k},\ \text{where } e\cong 2.718.$$

EXAMPLE 47 For $f(x)=\dfrac{1}{x}$ and $g(x) = x$, find $\lim\limits_{x\to\infty}(1+f(x))^{g(x)}$.

Solution We have $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\dfrac{1}{x}=0$,

$\lim\limits_{x\to\infty} g(x)=\lim\limits_{x\to\infty} x=\infty$ and

$$\lim_{x\to\infty} f(x)\cdot g(x)=\lim_{x\to\infty}\left(\frac{1}{x}\cdot x\right)=1.$$

So $k = 1$ and by the given rule,

$$\lim_{x\to\infty}(1+f(x))^{g(x)}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e^{1}=e.$$

EXAMPLE 48 For $f(x)=\dfrac{3}{x}$ and $g(x) = 5x$, find $\lim\limits_{x\to\infty}(1+f(x))^{g(x)}$.

Solution We have $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\dfrac{3}{x}=0$,

$\lim\limits_{x\to\infty} g(x)=\lim\limits_{x\to\infty} 5x=\infty$ and

$\lim\limits_{x\to\infty} f(x)\cdot g(x)=\lim\limits_{x\to\infty}\left(\dfrac{3}{x}\cdot 5x\right)=15$. So $k = 15$ and by the given rule,

$$\lim_{x\to\infty}(1+f(x))^{g(x)}=\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{5x}=e^{15}.$$

> For $m, n \in \mathbb{R}$,
> $\lim\limits_{x\to\infty}(1+\dfrac{m}{x})^{nx}=e^{mn}$.

EXAMPLE 49 Find $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{2x+1}\right)^{3x+1}$.

Solution We can rewrite the limit in the form $\lim\limits_{x\to\infty}(1+f(x))^{g(x)}$ using $f(x)=\dfrac{2}{2x+1}$ and $g(x) = 3x + 1$.

So $f(x)=\dfrac{2}{2x+1}$ and $\lim\limits_{x\to\infty} f(x)=\lim\limits_{x\to\infty}\dfrac{2}{2x+1}=0$,

$g(x)=3x+1$ and $\lim\limits_{x\to\infty} g(x)=\lim\limits_{x\to\infty} 3x+1=\infty$.

This gives $\lim\limits_{x\to\infty} f(x)\cdot g(x)=\lim\limits_{x\to\infty}\dfrac{2}{2x+1}\cdot(3x+1)=\lim\limits_{x\to\infty}\dfrac{6x+2}{2x+1}=\dfrac{6}{2}=3$. So $k = 3$ and

by the given rule, $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{2x+1}\right)^{3x+1}=e^{3}$.

EXAMPLE 50 Find $\lim_{x\to\infty}\left(\frac{x-1}{x}\right)^x$.

Solution We can rewrite $\frac{x-1}{x}$ as $1-\frac{1}{x}$.

Then we can take $f(x)=-\frac{1}{x}$ and $g(x)=x$. As $x\to\infty$, $f(x)\to 0$ and $g(x)\to\infty$.

Since $\lim_{x\to\infty} f(x)\cdot g(x)=\lim_{x\to\infty}\left(-\frac{1}{x}\right)\cdot(x)=-1$, $k=-1$.

So $\lim_{x\to\infty}\left(\frac{x-1}{x}\right)^x=\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x=e^{-1}=\frac{1}{e}$.

EXAMPLE 51 Find $\lim_{x\to\infty}\left(\frac{3x+1}{3x-1}\right)^{4x-2}$.

Solution We can write $\frac{3x+1}{3x-1}=\frac{3x-1+2}{3x-1}=1+\frac{2}{3x-1}$.

Then $\lim_{x\to\infty}\left(\frac{3x+1}{3x-1}\right)^{4x-2}=\lim_{x\to\infty}\left(1+\frac{2}{3x-1}\right)^{4x-2}$ and we can take $f(x)=\frac{2}{3x-1}$ and $g(x)=4x-2$.

Since $\lim_{x\to\infty} f(x)\cdot g(x)=\lim_{x\to\infty}\left(\frac{2}{3x-1}\right)\cdot(4x-2)=\lim_{x\to\infty}\frac{8x-4}{3x-1}=\frac{8}{3}$, $k=\frac{8}{3}$.

So $\lim_{x\to\infty}\left(\frac{3x+1}{3x-1}\right)^{4x-2}=\lim_{x\to\infty}\left(1+\frac{2}{3x-1}\right)^{4x-2}=e^{\frac{8}{3}}$.

Check Yourself 14

Calculate the limits.

1. $\lim_{x\to\infty}\left(1+\frac{5}{x}\right)^x$
2. $\lim_{x\to\infty}\left(\frac{x-2}{x}\right)^x$
3. $\lim_{x\to\infty}\left(\frac{2x+1}{2x}\right)^x$
4. $\lim_{x\to\infty}\left(\frac{3x-2}{3x}\right)^{4x}$
5. $\lim_{x\to\infty}\left(1-\frac{3}{2x+7}\right)^{2x+5}$
6. $\lim_{x\to\infty}\left(\frac{2x-5}{2x+3}\right)^{x+4}$
7. $\lim_{x\to\infty}\left(\frac{x+2}{x+5}\right)^{3x+1}$

Answers

1. e^5 2. e^{-2} 3. $e^{\frac{1}{2}}$ 4. $e^{-\frac{8}{3}}$ 5. e^{-3} 6. e^{-4} 7. e^{-9}

EXERCISES 2.2

A. $\frac{0}{0}$ as a Limit

1. Calculate the limits.

a. $\lim\limits_{x \to 5} \dfrac{x-5}{x^2-25}$

b. $\lim\limits_{x \to -2} \dfrac{x^2-x-6}{x^2+x-2}$

c. $\lim\limits_{x \to -1} \dfrac{x^2-5x-6}{x+1}$

d. $\lim\limits_{x \to 1} \dfrac{5x^3-5}{x-1}$

e. $\lim\limits_{x \to 3} \dfrac{x^2-9}{x^2-5x+6}$

f. $\lim\limits_{x \to 2} \dfrac{x-2}{2x-\sqrt{3x^2+4}}$

g. $\lim\limits_{x \to -2} \dfrac{2-|x|}{2+x}$

h. $\lim\limits_{x \to m} \dfrac{x^3-m^3}{2x^2-mx-m^2}$

i. $\lim\limits_{x \to 0} \dfrac{2^{2x+4}-16}{2^{x+1}-2}$

2. Calculate the limits.

a. $\lim\limits_{x \to 0} \dfrac{\sin 5x}{7x}$

b. $\lim\limits_{x \to 3} \dfrac{x-3}{\tan(2x-6)}$

c. $\lim\limits_{x \to 2} \dfrac{\sin(3x-6)}{\tan(x-2)}$

d. $\lim\limits_{x \to 0} \dfrac{\sqrt{1-\cos 2x}}{\sin 2x}$

e. $\lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{1-\cos 2x}}$

f. $\lim\limits_{x \to \pi} \dfrac{\cos x+1}{(\pi-x)^2}$

g. $\lim\limits_{x \to 0} \dfrac{\cos^2 x-1}{x^2}$

h. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\pi-2x}{\sin 2x}$

i. $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{\cos x-\sqrt{\cos 2x}}$

j. $\lim\limits_{x \to 1} \dfrac{(1-x^2)(x^3-1)}{\tan^2(x-1)}$

B. $\frac{\infty}{\infty}$ as a Limit

3. Calculate the limits.

a. $\lim\limits_{x \to \infty} \dfrac{3x^3+2x^2+1}{x^3-1}$

b. $\lim\limits_{x \to \infty} \dfrac{x^2+x+1}{3x^3+4}$

c. $\lim\limits_{x \to -\infty} \dfrac{x^2+1}{5x-4}$

d. $\lim\limits_{x \to -\infty} \dfrac{x^2+x+2}{3+\sqrt[5]{x^5+1}}$

e. $\lim\limits_{x \to \infty} \dfrac{2005x+2010}{x^2+1}$

f. $\lim\limits_{x \to \infty} \dfrac{x^4+ax+2}{(x^2-1)^2-a}$

g. $\lim\limits_{x \to -\infty} \dfrac{|x|+2x-1}{3x+4|x|+5}$

h. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2+x-2}+3x}{\sqrt[5]{x^5}-1}$

i. $\lim\limits_{x \to \infty} \log\left(\dfrac{x^2+2x+1}{x^2-7}\right)$

j. $\lim\limits_{x \to \infty} \dfrac{2x}{\cot \dfrac{2}{x}}$

k. $\lim\limits_{x \to \infty} \dfrac{2x}{\sqrt{\dfrac{x^3}{3x-1}}}$

l. $\lim\limits_{x \to -\infty} \dfrac{||x|-x+3|}{4|x|}$

C. $0 \cdot \infty$ as a Limit

4. Calculate the limits.

a. $\lim\limits_{x \to \infty}\left(x \cdot \sin \dfrac{6}{7x}\right)$

b. $\lim\limits_{x \to \infty} \sqrt{x^2-x+3} \cdot \dfrac{5}{x+2}$

c. $\lim\limits_{x \to \infty}\left(\sqrt{3}x \cdot \tan \dfrac{\sqrt{3}}{x}\right)$

d. $\lim\limits_{x \to 0}(\sin 2x \cdot \cot 5x)$

e. $\lim\limits_{x \to 0}(\tan \sqrt{x} \cdot \cot 3\sqrt{x})$

f. $\lim\limits_{x \to 1}\left(\tan(1-x) \cdot \tan \dfrac{\pi x}{2}\right)$

D. $\infty - \infty$ as a Limit

5. Calculate the limits.

a. $\lim\limits_{x \to \infty}\left(\dfrac{x^2}{3x+4}-\dfrac{x}{3}\right)$

b. $\lim\limits_{x \to -\infty}\left(\dfrac{2x^3-x^2}{1+2x^2}-x\right)$

c. $\lim\limits_{x \to 0}\left(\dfrac{x+2}{x}-\dfrac{4}{x^2}\right)$

d. $\lim\limits_{x \to 1}\left(\dfrac{x}{x-1}+\dfrac{2}{x^2-4x+3}\right)$

e. $\lim\limits_{x \to \frac{\pi}{2}}\left(\dfrac{1}{\cos x}-\dfrac{1}{\cot x}\right)$

f. $\lim\limits_{x \to 0}(\cot x-\operatorname{cosec} x)$

6. Calculate the limits.

a. $\lim\limits_{x\to\infty}(\sqrt{x^2+5x+1}-x)$

b. $\lim\limits_{x\to\infty}(\sqrt{x^2+x+1}-\sqrt{x^2+2x-3})$

c. $\lim\limits_{x\to\infty}(\sqrt{x^2+2x+2}-x+1)$

d. $\lim\limits_{x\to\infty}(x\sqrt{x^2+1}-x^2)$

e. $\lim\limits_{x\to-\infty}(x\sqrt{x^2-2}+x^2)$

f. $\lim\limits_{x\to-\infty}(\sqrt{x^2-1}-\sqrt{x^2+1})$

E. 1^{∞} as a Limit

7. Calculate the limits.

a. $\lim\limits_{x\to\infty}(1+\frac{2}{x})^x$

b. $\lim\limits_{x\to\infty}(1-\frac{3}{x})^{3x}$

c. $\lim\limits_{x\to\infty}(\frac{x+3}{x})^x$

d. $\lim\limits_{x\to\infty}(\frac{6x+1}{6x})^{2x}$

e. $\lim\limits_{x\to\infty}(1-\frac{2}{3x+1})^{2x+1}$

f. $\lim\limits_{x\to\infty}(\frac{5x+1}{5x-1})^{3x+1}$

g. $\lim\limits_{x\to\infty}(1+\frac{7}{5x+3})^{2x}$

h. $\lim\limits_{x\to\infty}(\frac{x+2}{x+4})^{9x-1}$

i. $\lim\limits_{x\to\infty}(\frac{3x+1}{3x+2})^{3x+1}$

j. $\lim\limits_{x\to\infty}(\frac{2x+7}{2x-1})^{x-1}$

Mixed Problems

8. Calculate the limits.

a. $\lim\limits_{x\to\infty}(2+2x\sin\frac{4}{x})$

b. $\lim\limits_{x\to\frac{\pi}{2}}((1-\sin x)\cdot\tan x)$

c. $\lim\limits_{x\to 0}((x^2+\sin^2 x)\cdot\cot^2 x)$

d. $\lim\limits_{x\to\infty}(\sqrt{\frac{x}{2}}\cdot\sin\frac{8}{x})$

e. $\lim\limits_{x\to 0}(\sqrt{x}(\sqrt{x+2}-\sqrt{x}))$

f. $\lim\limits_{x\to\infty}(\log_2\sqrt{x^2+x}-\log_2\sqrt{x^2-3x})$

g. $\lim\limits_{x\to\frac{\pi}{2}}(\frac{1}{1-\sin x}-\frac{2}{\cos^2 x})$

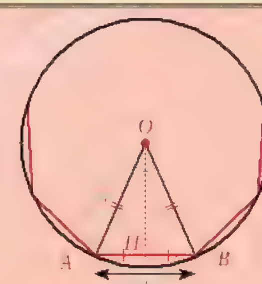
h. $\lim\limits_{x\to\infty}(\log_2\sqrt{8x+1}+\log_{1/2}\sqrt{2x+5})$

9. Calculate the limits.

a. $\lim\limits_{x\to 2}(\frac{5x-1}{|2-x|}-\frac{1}{x^2-4})$

b. $\lim\limits_{x\to 3^-}(\frac{[\![x-3]\!]}{x-3}-\frac{1}{|x-3|})$

Finding the Circumference of a Circle by Using a Limit



The figure shows an n-sided regular polygon inside a circle centered at O.

In the figure let $|AB| = a$, $|OA| = r$ and $m(\angle AOB)=\frac{2\pi}{n}$ so in the right triangle AOH,

$\sin(\angle AOH)=\frac{\frac{a}{2}}{r}=\frac{a}{2r}$ and $a = 2r\sin(\angle AOH)$.

Since ΔAOB is an isosceles triangle, $m(\angle AOH)=\frac{\pi}{n}$ and $a=2r\sin\frac{\pi}{n}$.

The perimeter of the polygon is therefore $n\cdot 2r\sin\frac{\pi}{n}$.

As the number of sides of the regular polygon increases to infinity, the polygon gets closer and closer to a circle.

Therefore the circumference of the circle is $\lim\limits_{n\to\infty} n\cdot 2r\sin\frac{\pi}{n}=\lim\limits_{n\to\infty} 2r\,\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\cdot\pi=2\pi r$.

It can also be proved that the area of a circle is equal to πr^2 by using a limit. (This is left as an exercise for you.)

CHAPTER SUMMARY

- The limit of a polynomial function $f(x)$ as x approaches a point c is $f(c)$.
- The interval $(x-\varepsilon, x+\varepsilon)$ is called the ε-neighborhood of x.
- Given any ε about L if there exists a δ about x_0 such that for all x, $|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$ then the limit of f as x approaches the point x_0 is L, and it is denoted by $\lim\limits_{x\to x_0} f(x)=L$.
- A function $f(x)$ has a limit at a point x_0 if and only if the right-hand and left-hand limits at x_0 exist and are equal.

  $\lim\limits_{x\to x_0} f(x)=L \Leftrightarrow \lim\limits_{x\to x_0^+} f(x)=L$ and $\lim\limits_{x\to x_0^-} f(x)=L$
- For $h\in\mathbb{R}^+$ and $h\to 0$, the following limits are equal:

  $\lim\limits_{x\to a^+} f(x)=\lim\limits_{h\to 0} f(a+h)$ and $\lim\limits_{x\to a^-} f(x)=\lim\limits_{h\to 0} f(a-h)$
- $\lim\limits_{x\to x_0} f(x)=\infty$ means that for any chosen number M, we can always find an x closer to x_0 such that $f(x)>M$
- $\lim\limits_{x\to x_0} f(x)=-\infty$ means that for any chosen number $-M$, we can always find an x closer to x_0 such that $f(x)<-M$.
- $\lim\limits_{x\to +\infty} f(x)=c$ means that for any chosen $\varepsilon>0$, we can find a number M such that for all $x>M$, the value of $f(x)$ will be in the ε-neighborhood of c, i.e. $f(x)$ gets closer and closer to c as x approaches positive infinity.
- $\lim\limits_{x\to -\infty} f(x)=c$ means that for any chosen $\varepsilon>0$, we can find a number N such that for all $x<N$, the value of $f(x)$ will be in the ε-neighborhood of c, i.e. $f(x)$ approaches the number c as x approaches negative infinity.
- If $f(x)$ and $g(x)$ are functions such that $\lim\limits_{x\to x_0} f(x)=a$ and $\lim\limits_{x\to x_0} g(x)=b$ then

  1. $\lim\limits_{x\to x_0}[f(x)+g(x)]=a+b$
  2. $\lim\limits_{x\to x_0}[f(x)-g(x)]=a-b$
  3. $\lim\limits_{x\to x_0}[f(x)\cdot g(x)]=a\cdot b$
  4. $\lim\limits_{x\to x_0}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{a}{b},\ (b\neq 0)$
  5. $\lim\limits_{x\to x_0} k\cdot f(x)=k\cdot a \quad (k\in\mathbb{R})$.
- The numbers $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$ and 1^∞ are called indeterminate forms. They can be removed from a limit with some simple substitutions:
- 1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ and $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$.

  2. $\lim\limits_{x\to\pm\infty}\dfrac{P(x)}{Q(x)}=\lim\limits_{x\to\pm\infty}\dfrac{a_n x^n}{b_m x^m}=\begin{cases}\dfrac{a_n}{b_m} & \text{if } n=m\\ 0 & \text{if } n<m\\ \pm\infty & \text{if } n>m\end{cases}$

  3. Given $f(x)=\sqrt{ax^2+bx+c}$, $\quad (a>0)$

  $$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\sqrt{ax^2+bx+c}=\lim_{x\to\pm\infty}\sqrt{a}\left|x+\frac{b}{2a}\right|$$

  4. Let $\lim\limits_{x\to x_0} f(x)=0$, $\lim\limits_{x\to x_0} g(x)=\infty$, and

  $\lim\limits_{x\to x_0} f(x)\cdot g(x)=k\ (k\in\mathbb{R})$ Then $\lim\limits_{x\to x_0}(1+f(x))^{g(x)}=e^k$,

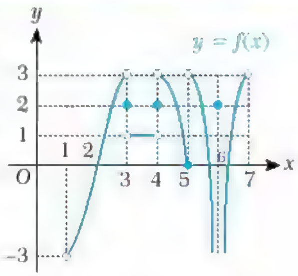
  where $e\cong 2.718$.

Concept Check

1. Explain the concept of limit in your own words.
2. When calculating a limit at a given point, is it important what happens at that point?
3. What is the ε-neighborhood of a point x?
4. What is the mathematical definition of the limit of a function?
5. What is the relation between ε and δ in the definition of limit?
6. Explain the concept of one-sided limit.
7. What is the importance of the equivalence of one sided-limits?
8. Explain the importance of one-sided limits when evaluating the limit of an absolute value function.
9. How can we rewrite limits as $x\to x_0^+$ and $x\to x_0^-$ using the very small positive number h?
10. If $f(x)$ remains in the ε-neighborhood of a number c as x approaches infinity, what is the limit as x approaches infinity?
11. How many different types of indeterminate form are there?
12. How can we remove the indeterminate form $\frac{0}{0}$ from a limit?
13. How can we remove the indeterminate form $\frac{\infty}{\infty}$ from a limit?
14. How can we remove the indeterminate form $0\cdot\infty$?
15. How can we remove the indeterminate form $\infty-\infty$?
16. How can we remove the indeterminate form 1^∞?

CHAPTER REVIEW TEST 2A

1. The figure shows the graph of $f(x)$. At which integer point in the interval $(1, 7)$ does a finite limit of $f(x)$ exist?



A) 2 B) 3 C) 4 D) 5 E) 6

2. Find $\lim\limits_{x\to-\infty}(x^7+x^6-x^5)$.

A) $-\infty$ B) ∞ C) 0 D) 2 E) –1

3. Find $\lim\limits_{x\to0^-}\dfrac{1}{x}$.

A) 0 B) 1 C) –1 D) ∞ E) $-\infty$

4. Find $\lim\limits_{x\to-\infty}\dfrac{2x^3-x^2-1}{x^2+2x+1}$.

A) ∞ B) $-\infty$ C) 2 D) –2 E) 0

5. Find $\lim\limits_{x\to0^-}\dfrac{3x+|x|}{3x+2|x|}$.

A) 0 B) 1 C) 2 D) 3 E) 4

6. Find $\lim\limits_{x\to2^+}\dfrac{|2-x|}{x^2-\operatorname{sgn}(x-2)}$.

A) 0 B) 1 C) 2 D) 3 E) 4

7. Find $\lim\limits_{x\to0}\dfrac{\sqrt{4+3x}-2}{x}$.

A) 0 B) $\dfrac{1}{4}$ C) $\dfrac{3}{2}$ D) $\dfrac{3}{4}$ E) $\dfrac{4}{3}$

8. Find $\lim\limits_{x\to\infty}\dfrac{x^2+\cos x}{3x^2}$.

A) $-\dfrac{1}{3}$ B) $\dfrac{1}{3}$ C) –1 D) 1 E) 0

9. Find $\lim_{x\to\infty}(x \cdot \sin\frac{6}{7x})$.

A) $\frac{7}{6}$ B) $-\frac{6}{7}$ C) $\frac{6}{7}$ D) 0 E) ∞

10. Find $\lim_{x\to\infty}(1+\frac{5}{x})^{3x}$.

A) e^{-3} B) e^{3} C) e^{5} D) e^{-15} E) e^{15}

11. Find $\lim_{x\to 0}(\frac{\sin 4x}{x})^{x+2}$.

A) 2 B) 4 C) 6 D) 8 E) 16

12. Find $\lim_{x\to\infty}(\sqrt{x^2-3x+4}-x)$.

A) $\frac{1}{2}$ B) $-\frac{3}{2}$ C) $\frac{3}{2}$ D) $-\frac{1}{2}$ E) 0

13. Find $\lim_{x\to a}(\frac{a^3-x^3}{a^2-x^2})$.

A) $\frac{3a}{2}$ B) $\frac{a}{2}$ C) $\frac{3}{2}$ D) $3a$ E) $\frac{1}{2a}$

14. Find $\lim_{x\to 0}(\frac{3^x-3^{-x}}{3^x+3^{-x}})$.

A) ∞ B) $-\infty$ C) 0 D) 1 E) 2

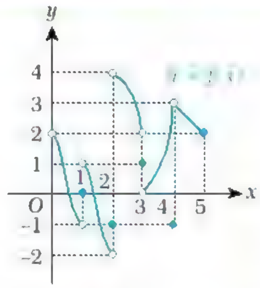
15. Find $\lim_{x\to -1}(3x+\frac{|x+1|}{x+1})$.

A) 2 B) –2 C) 4 D) –4 E) does not exist

16. Find $\lim_{x\to 0}(\frac{\tan^2 x}{x})$.

A) 0 B) 1 C) –1 D) ∞ E) $-\infty$

1. The graph of a function $f(x)$ is shown in the figure. Find the sum of the left-hand limits of the function at the integer values in the interval $(0, 5]$.



A) -1 B) 1 C) 2 D) 3 E) 4

2. Find $\lim\limits_{x\to\infty}(1-x-x^2)$.

A) -1 B) 1 C) 0 D) ∞ E) $-\infty$

3. Find $\lim\limits_{x\to 9}\dfrac{x-9}{\sqrt{x}-3}$.

A) 0 B) 3 C) 6 D) 9 E) ∞

4. Find $\lim\limits_{x\to\infty}(\sqrt{x^2+x+1}-x+7)$.

A) $-\dfrac{15}{2}$ B) $\dfrac{15}{2}$ C) $\dfrac{7}{2}$ D) $-\dfrac{7}{2}$ E) 7

5. Find $\lim\limits_{x\to\pi^+}\dfrac{[\![\sin x]\!]}{\sin x}$.

A) $-\infty$ B) ∞ C) 0 D) 1 E) -1

6. Find $\lim\limits_{x\to 2^-}\dfrac{x^2-2x}{x-3\,\mathrm{sgn}(x-2)}$.

A) -1 B) 1 C) 0 D) ∞ E) $-\infty$

7. Find $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\mathrm{sgn}(\sin 2x)}{\mathrm{sgn}(\cos 3x)}$.

A) $-\infty$ B) ∞ C) 0 D) -1 E) 1

8. Find $\lim\limits_{x\to\infty}\dfrac{12x^3+2x^2-1}{x^3+1}$.

A) 6 B) -6 C) 12 D) -12 E) 14

9. Find $\lim\limits_{x\to\infty}(e^{x}\cdot\tan e^{-x})$.

A) ∞ B) $-\infty$ C) 0 D) 1 E) -1

10. Find $\lim\limits_{x\to\infty}\dfrac{(x+2)(3x-1)}{5x^2-4}$.

A) $\dfrac{6}{5}$ B) $\dfrac{2}{5}$ C) $-\dfrac{1}{4}$ D) $-\dfrac{1}{5}$ E) $\dfrac{3}{5}$

11. Find $\lim\limits_{x\to\infty}(\dfrac{x+5}{x})^{2x}$.

A) e^5 B) e^{10} C) e^2 D) e E) 1

12. Find $\lim\limits_{x\to 0}\dfrac{\sin^3 2x}{x^3}$.

A) 0 B) 1 C) 2 D) 4 E) 8

13. Find $\lim\limits_{x\to 2}(\dfrac{x^3-8x+8}{x^4-4x})$.

A) 0 B) 2 C) 4 D) 8 E) 16

14. $f(x)=\begin{cases}\dfrac{1}{|x|} & \text{if } x<0\\ \ln x & \text{if } x>0\end{cases}$ is given. Find $\lim\limits_{x\to 0}f(x)$.

A) $-\infty$ B) ∞ C) 0 D) 1 E) does not exist

15. Find $\lim\limits_{x\to 1}(\dfrac{1}{1-x}-\dfrac{2}{1-x^2})$.

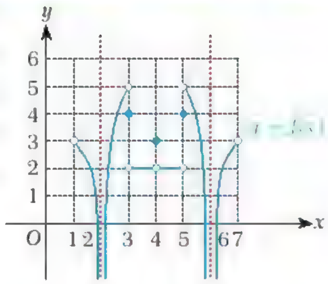
A) -1 B) 1 C) $-\dfrac{1}{2}$ D) $\dfrac{1}{2}$ E) 0

16. Find $\lim\limits_{x\to\infty}(\log_5\sqrt{\dfrac{50x^3-x^2+1}{2x^3+5}})$.

A) 2 B) -2 C) -1 D) 1 E) 0

CHAPTER REVIEW TEST 2C

1. The figure shows the graph of a function $f(x)$. At which integer point in the interval $(1,\ 7)$ does a finite limit of $f(x)$ exist?



A) 2 B) 3 C) 4 D) 5 E) 6

2. Find $\lim\limits_{x \to 2} \dfrac{5x-1}{|2-x|}$.

A) $-\infty$ B) ∞ C) 1 D) -1 E) does not exist

3. Find $\lim\limits_{x \to \frac{\pi}{2}^-} (x - \tan x)$.

A) 0 B) 1 C) -1 D) ∞ E) $-\infty$

4. Find $\lim\limits_{x \to 2^-} ([\![x]\!] + 3\operatorname{sgn} x)$.

A) 1 B) 2 C) 3 D) 4 E) 5

5. Find $\lim\limits_{x \to 1^+} \left(\dfrac{|1-x|}{x^2-1} - [\![x-1]\!]\right)$.

A) $-\dfrac{1}{2}$ B) $\dfrac{1}{2}$ C) 0 D) 1 E) -1

6. Find $\lim\limits_{x \to \pi^-} [\![1 - \cos x]\!]$.

A) -1 B) 0 C) 1 D) 2 E) 3

7. Find $\lim\limits_{x \to \frac{7\pi}{4}} \dfrac{\sin x + \cos x}{\cos 2x}$.

A) $\dfrac{\sqrt{2}}{2}$ B) $\sqrt{2}$ C) $-\dfrac{\sqrt{2}}{2}$ D) $\dfrac{\sqrt{2}}{3}$ E) $-\dfrac{\sqrt{2}}{3}$

8. Find $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x}$.

A) 1 B) -1 C) 0 D) ∞ E) $-\infty$

9. Find $\lim\limits_{x\to\infty}\left(\dfrac{2x+3}{2x-3}\right)^{4x}$.

A) e^{6} B) e^{12} C) e^{18} D) e^{24} E) e^{30}

10. Find $\lim\limits_{x\to\infty}\dfrac{(x+1)^3(2x-1)^2}{5x^5-3}$.

A) $-\dfrac{1}{3}$ B) $\dfrac{1}{3}$ C) $-\dfrac{1}{5}$ D) $\dfrac{4}{5}$ E) $\dfrac{2}{5}$

11. Find $\lim\limits_{x\to-\infty}\dfrac{2x-\sqrt{x^2-4x+1}}{x-\sqrt[3]{1-x^3}}$.

A) $\dfrac{3}{2}$ B) $\dfrac{2}{3}$ C) $-\dfrac{3}{2}$ D) $-\dfrac{2}{3}$ E) 0

12. Find $\lim\limits_{x\to 0}\left(\dfrac{\tan 2x}{x}\right)^{\frac{\sin 3x}{x}}$.

A) 4 B) 8 C) 10 D) 5 E) 2

13. Find $\lim\limits_{x\to 3}\dfrac{x^2-9}{2-\sqrt{x+1}}$.

A) 1 B) 10 C) –10 D) –24 E) $-\infty$

14. Find $\lim\limits_{x\to-\infty}(2x-1+\sqrt{4x^2+x-3})$.

A) $\dfrac{3}{5}$ B) $\dfrac{4}{5}$ C) $-\dfrac{4}{5}$ D) $\dfrac{5}{4}$ E) $-\dfrac{5}{4}$

15. Find $\lim\limits_{x\to-\frac{\pi}{4}^-}\dfrac{\tan(x+\frac{3\pi}{4})}{\tan^2 x}$.

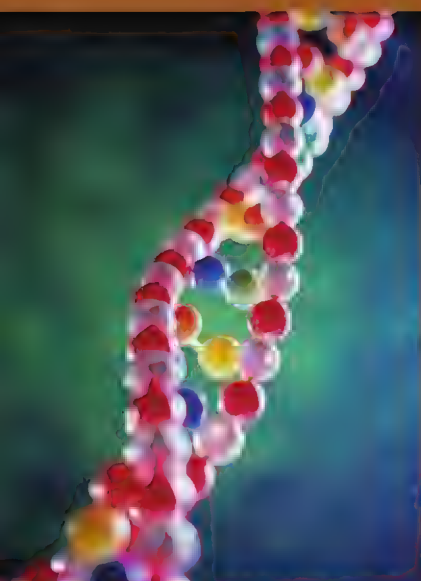
A) $-\infty$ B) ∞ C) 0 D) 1 E) –1

16. Find $\lim\limits_{x\to-\frac{\pi}{4}^-}\dfrac{\sqrt{1+\cos 4x}}{\sin 4x}$.

A) –1 B) 1 C) 0 D) $\dfrac{\sqrt{2}}{2}$ E) $-\dfrac{\sqrt{2}}{2}$

CHAPTER 3

CONTINUITY



1 CONTINUOUS FUNCTIONS

A. CONTINUITY AT A POINT

In this section we will look at the concept of continuity of a function. During our study of the limit of $f(x)$ as x approaches x_0, we have continuously emphasized that the limit is not necessarily equal to $f(x_0)$. Indeed, what actually happens at the point x_0 is not important for the limit of the function at that point. However, the nature of $f(x_0)$ becomes important when we are considering the continuity of a function.

Definition

continuity at a point, discontinuity at a point

Let A be a subset of $\mathbb{R}$, let $f: A \to \mathbb{R}$ be a function and let $x_0 \in A$. If

$$\lim_{x \to x_0} f(x) = f(x_0)$$

then we say f is continuous at x_0. Otherwise, f is said to be discontinuous at x_0.

It is important to note that for a function f to be continuous at x_0, three things are necessary:

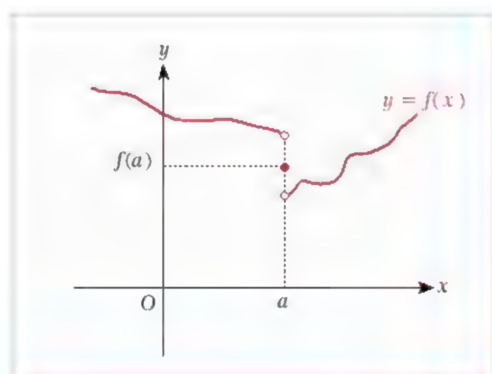
1. The limit of $f(x)$ as $x \to x_0$ must exist.
2. The function f must be defined at the number x_0, i.e. $f(x_0)$ must exist.

And also, by the definition above,

3. $\lim_{x \to x_0} f(x) = f(x_0)$.

If even one of these three conditions is not satisfied, the function f is said to be discontinuous at point x_0.

Let us begin by looking at the continuity of some functions at given points.



The figure shows the graph $y = f(x)$.

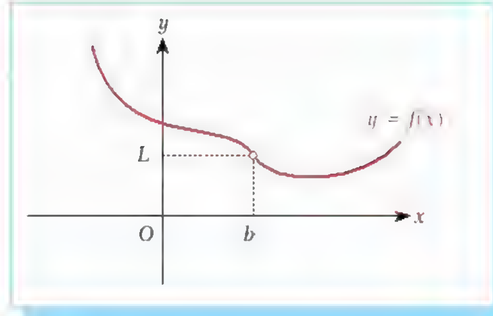
At $x = a$, $\lim_{x \to a} f(x)$ does not exist.

So f is discontinuous at $x = a$.

At $x = b$, $\lim\limits_{x \to b} f(x) = L$ but $f(b)$ does not exist.

This means that f is not defined at $x = b$.

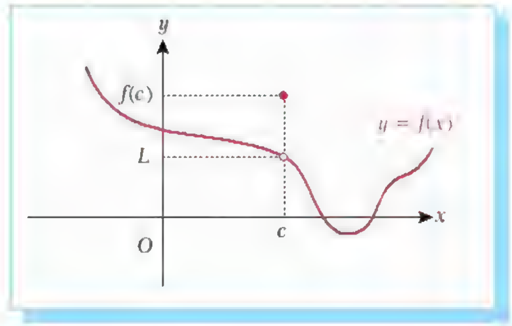
So f is discontinuous at point b.



At $x = c$, $\lim\limits_{x \to c} f(x) = L$ and $f(c)$ exists, but

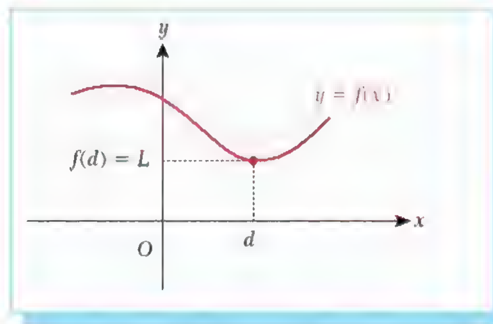
$$\lim_{x \to c} f(x) \neq f(c).$$

So f is discontinuous at point c.



At $x = d$, $\lim\limits_{x \to d} f(x) = f(d)$.

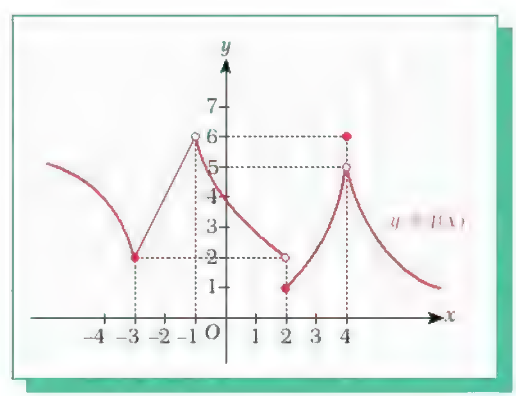
So f is continuous at point d.



Note

The graph of a function which is continuous at a point has no gaps or breaks in its line at that point. In other words, we can draw the graph of the function without lifting our pen from the paper when we pass through the point.

EXAMPLE 1 Examine the continuity of the function $f(x)$ in the figure at the points $x = -3$, $x = -1$, $x = 2$ and $x = 4$.



Solution a. The graph shows $f(-3) = 2$ and $\lim\limits_{x \to -3} f(x) = 2$, so

$\lim\limits_{x \to -3} f(x) = f(-3)$. So f is continuous at $x = -3$.

b. $f(-1)$ is not defined. So f is discontinuous at $x = -1$.

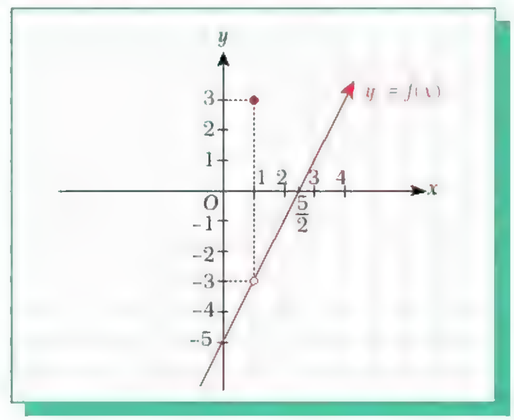
c. $\lim\limits_{x \to 2} f(x)$ does not exist. So f is discontinuous at $x = 2$.

d. $\lim\limits_{x \to 4} f(x) = 5$ but $f(4) = 6$.

Since $\lim\limits_{x \to 4} f(x) \neq f(4)$, f is discontinuous at $x = 4$.

EXAMPLE 2 Examine the continuity of f: $\mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x - 5 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases} \quad \text{at the point } x = 1.$$



Solution The graph of $f(x)$ is shown in the figure. Since $f(1) = 3$, $f(x)$ is defined at $x = 1$.

However, $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1}(2x - 5) = -3$.

We can see that $\lim\limits_{x \to 1} f(x) \neq f(1)$,

so f is discontinuous at $x = 1$.

EXAMPLE 3 Examine the continuity of f: $\mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ 2x & \text{if } x = 1 \\ -x + 3 & \text{if } x > 1 \end{cases} \quad \text{at the point } x = 1.$$

Solution Since $f(1) = 2 \cdot 1 = 2$, $f(x)$ is defined at $x = 1$.

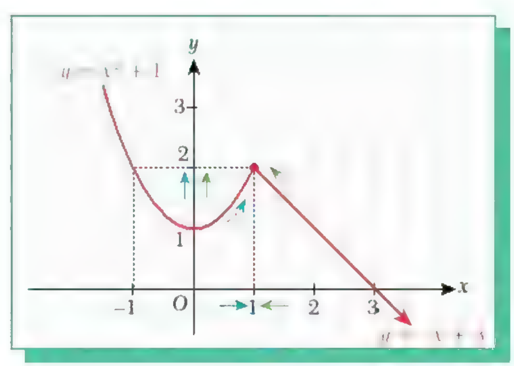
Let us find the limit of the function at $x = 1$:

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-}(x^2 + 1) = 1^2 + 1 = 2$ and

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+}(-x + 3) = -1 + 3 = 2$, so

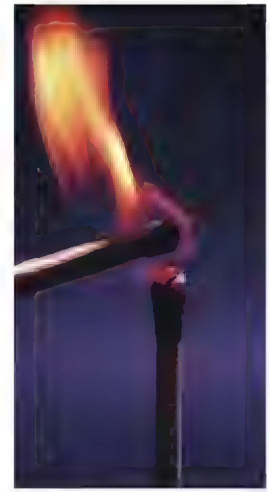
$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = 2$ and so $\lim\limits_{x \to 1} f(x) = 2$.

Since $\lim\limits_{x \to 1} f(x) = f(1)$, f is continuous at $x = 1$.



EXAMPLE **4** Examine the continuity of $f: \mathbb{R} - \{-2\} \to \mathbb{R}$,

$$f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4} & \text{if } x \neq 2 \text{ and } x \neq -2 \\ 3 & \text{if } x = 2 \end{cases}$$ at the points $x = 2$ and $x = -2$.



Solution a. At $x = 2$, $f(2) = 3$.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$$

$$= \lim_{x\to 2} \frac{x^2+2x+4}{x+2} = \frac{12}{4} = 3$$

Since $\lim\limits_{x\to 2} f(x) = f(2)$, f is continuous at $x = 2$.

b. At $x = -2$, $f(-2)$ is not defined, so f is discontinuous at $x = -2$.

EXAMPLE **5** $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 2ax + 4 & \text{if } x > 3 \\ x - 5 & \text{if } x = 3 \\ bx + a & \text{if } x < 3 \end{cases}$ is a function which is continuous at $x = 3$.

Find the values of a and b.

Solution Since f is continuous at $x = 3$, we know $\lim\limits_{x\to 3} f(x) = f(3)$.

We can also write $\lim\limits_{x\to 3^+} f(x) = f(3)$, i.e.

$2a \cdot 3 + 4 = 3 - 5$ which gives $6a + 4 = -2$. So $6a = -6$, i.e. $a = -1$.

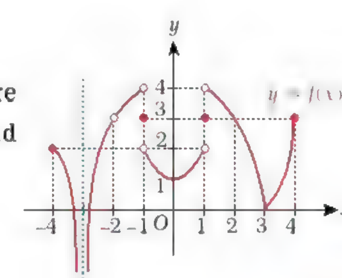
Similarly, we can write

$\lim\limits_{x\to 3^-} f(x) = f(3)$ which gives $b \cdot 3 - 1 = 3 - 5$; $3b - 1 = -2$; $3b = -1$ and thus $b = -\dfrac{1}{3}$.

So $a = -1$ and $b = -\dfrac{1}{3}$.

Check Yourself 1

1. Examine the continuity of the function f shown in the figure at the points $x = -3$, $x = -2$, $x = -1$, $x = 0$, $x = 2$ and $x = 3$.



2. Examine the continuity of $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \dfrac{8x-3}{3} & \text{if } x < -3 \\ 3x & \text{if } x = -3 \\ x-6 & \text{if } x > -3 \end{cases} \quad \text{at the point } x = -3.$$

3. Examine the continuity of $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^2 + 5x & \text{if } x < -1 \\ x^3 - 4x & \text{if } x \geq -1 \end{cases}$ at $x = -1$.

4. Examine the continuity of $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \dfrac{x-9}{x-3} & \text{if } x < 0 \\ 3 & \text{if } x = 0 \\ \dfrac{3}{\sqrt{1+x^2}} & \text{if } x > 0 \end{cases}$ at $x = 0$.

5. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} mx^2 + nx & \text{if } x < -2 \\ 8 & \text{if } x = -2 \\ \dfrac{mx-n}{\sqrt{5+x^2}} & \text{if } x > -2 \end{cases}$ is a function which is continuous at $x = -2$.

Find the values of m and n.

Answers

2. continuous 3. discontinuous 4. continuous 5. $m = -5$, $n = -14$

B. CONTINUITY ON AN INTERVAL

Theorem

If f and g are two continuous functions at a point $x = a$ then the following functions are also continuous at $x = a$:

1. $f + g$ 2. $f - g$ 3. $c \cdot f \ (c \in \mathbb{R})$ 4. $f \cdot g$ 5. $\dfrac{f}{g} \ (g(a) \neq 0)$.

Proof Let us prove that $f + g$ is a continuous function at $x = a$.

Since both f and g are continuous at $x = a$, both $f(a)$ and $g(a)$ are defined.

So $(f + g)(a) = f(a) + g(a)$ is also defined.

Furthermore, since $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist,

we can write
$$\begin{aligned}\lim_{x \to a}(f + g)(x) &= \lim_{x \to a}[f(x) + g(x)] \\ &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \\ &= f(a) + g(a) \\ &= (f + g)(a).\end{aligned}$$

Since $f + g$ satisfies the definition of continuity at $x = a$, it is a continuous function at this point.

Parts 2 to 5 of the theorem can be proved in a similar way. Their proof is left as an exercise for you.

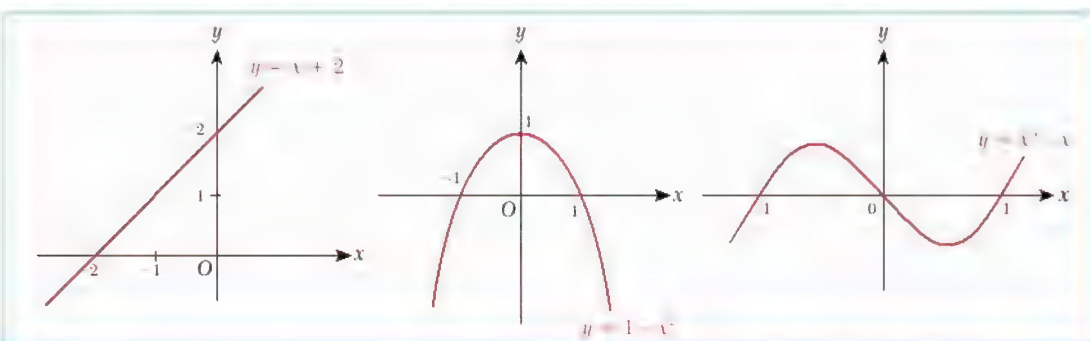
Definition **continuous function**

Let A be a subset of $\mathbb{R}$ and let $f: A \to \mathbb{R}$ be a function. If f is continuous at all $x \in A$ then it is said to be a continuous function in the domain A, or simply a continuous function.

Let us look at some examples of continuous functions.

1 Any polynomial function $P(x)$ is defined for all $x \in \mathbb{R}$. We also know that $\lim\limits_{x \to x_0} P(x) = P(x_0)$ for any polynomial function. So $P(x)$ is continuous at every point in the interval $(-\infty, +\infty)$. For example, the polynomial function $f(x) = 2x^3 - x^2 + 5$ is continuous at all $x \in \mathbb{R}$, so it is continuous in this domain.

Look at the graphs of some more polynomial functions:

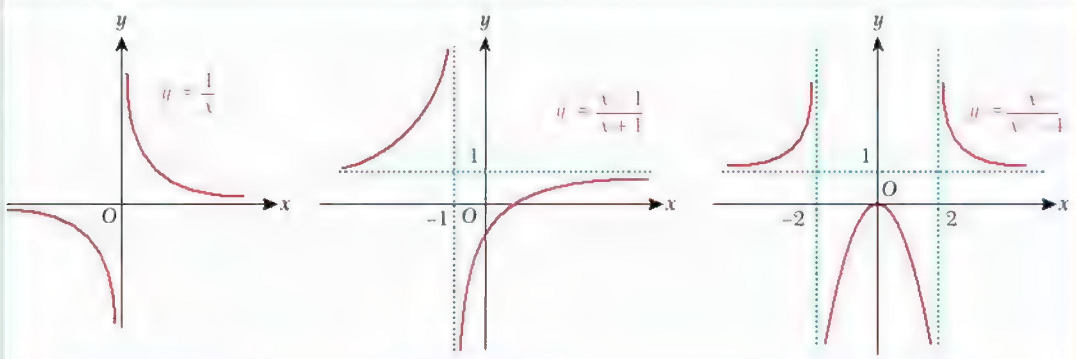


For all $x \in \mathbb{R}$, these polynomial functions are continuous. We can see that each graph is one continuous line: we can draw it without lifting our pen from the paper. This is true for all continuous functions.

A rational expression $f(x) = \dfrac{P(x)}{Q(x)}$ is continuous at all points in the interval $(-\infty, +\infty)$, provided $Q(x) \neq 0$.

For example, the rational function $g(x) = \dfrac{x^3 - 2x + 1}{x - 5}$ is continuous at all $x \in \mathbb{R}$ except $x = 5$. Similarly, the function $t(x) = (g(x))^5 = \left(\dfrac{x^3 - 2x + 1}{x - 5}\right)^5$ is continuous at all $x \in \mathbb{R}$ except $x = 5$.

Look at some more examples:



f is continuous for all $x \in \mathbb{R} - \{0\}$

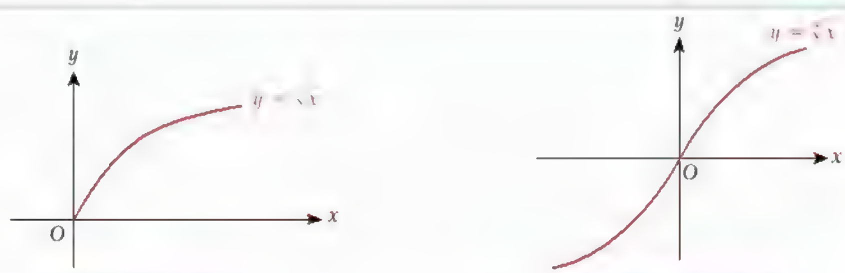
f is continuous for all $x \in \mathbb{R} - \{-1\}$

f is continuous for all $x \in \mathbb{R} - \{-2, 2\}$

For $n \in \mathbb{Z}^+$, the function $f(x) = \sqrt[2n]{P(x)}$ is defined when $P(x) \geq 0$, and $g(x) = \sqrt[2n+1]{P(x)}$ is defined for all $x \in \mathbb{R}$.

So *f* is continuous for $x \in \mathbb{R}$ satisfying $P(x) \geq 0$, and *g* is continuous for all $x \in \mathbb{R}$.

Look at some examples of this kind of function:

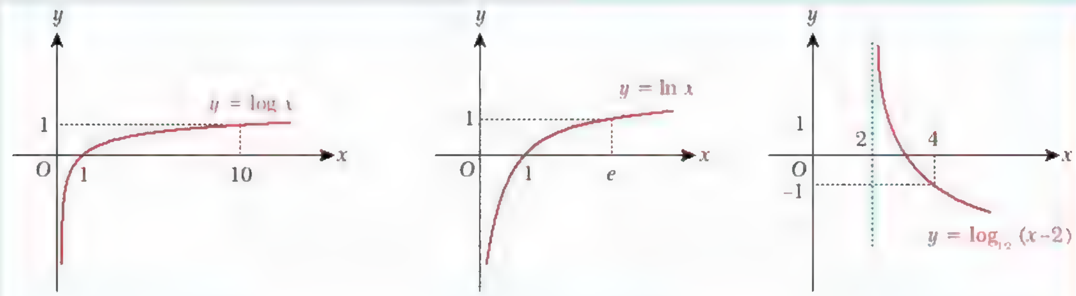


f is continuous for all $x \geq 0$

f is continuous for all $x \in \mathbb{R}$

For $a \in \mathbb{R}^+ - \{1\}$, the logarithmic function $f(x) = \log_a P(x)$ is continuous for all $x \in \mathbb{R}$ satisfying $P(x) > 0$.

Look at some examples of continuity for logarithmic functions:



f is continuous for all $x > 0$ | f is continuous for all $x > 0$ | f is continuous for all $x > 2$

Note

For $a \in \mathbb{R}^+ - \{1\}$, the exponential function $f(x) = a^x$ is continuous for $x \in \mathbb{R}$.

EXAMPLE 6 Find the largest interval on which the function $f(x) = \sqrt{9 - |x-1|}$ is continuous.

Solution f is continuous if $9 - |x - 1| \geq 0$, i.e.

$$|x - 1| \leq 9$$
$$-9 \leq x - 1 \leq 9$$
$$-8 \leq x \leq 10.$$

So f is continuous on the interval $[-8, 10]$.

EXAMPLE 7 Find the largest interval on which the function $f(x) = \dfrac{x+1}{x^2 - 5x + 6}$ is continuous.

Solution We know that a rational expression $f(x) = \dfrac{P(x)}{Q(x)}$ is continuous at all points provided that $Q(x) \neq 0$. In other words, f is discontinuous at the points which make the denominator zero. So we must find the roots of the denominator: $x^2 - 5x + 6 = 0$

$$(x-2)(x-3) = 0 \text{ i.e. } x_1 = 2 \text{ and } x_2 = 3.$$

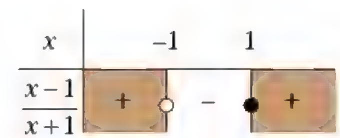
Hence f is continuous in the set $\mathbb{R} - \{2, 3\}$.

EXAMPLE 8 Find the largest interval on which the function $f(x)=\sqrt{\log\dfrac{2x}{x+1}}$ is continuous.

Solution We know that the logarithmic function is only defined for positive numbers, and for numbers between 0 and 1 it has negative values.

Since $\log\dfrac{2x}{x+1}$ must be positive, we need to find the solution of the inequality $\dfrac{2x}{x+1}\geq 1$:

$\dfrac{2x}{x+1}\geq 1$, $\dfrac{2x}{x+1}-1\geq 0$, so $\dfrac{x-1}{x+1}\geq 0$.



$x < -1$ and $x \geq 1$ are the solutions of the inequality.

Hence the function f is defined and continuous on the interval $(-\infty, -1) \cup [1, \infty)$.

EXAMPLE 9 For what values of x is the following function continuous?

$$f(x)=\begin{cases}\dfrac{x-4}{\sqrt{x}-2} & \text{if } x>4\\[2mm] 16-3x & \text{if } -4\leq x\leq 4\\[2mm] \dfrac{16}{x-4} & \text{if } x<-4\end{cases}$$

Solution Since the crucial points of the function are 4 and –4, let us examine the continuity of f at these points:

- At $x = 4$, $f(4) = 16 - 3\cdot 4 = 4$.

  Let us examine the one-sided limits at $x = 4$:

  $$\lim_{x\to 4^+} f(x)=\lim_{x\to 4^+}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4^+}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(\sqrt{x}-2)}=\lim_{x\to 4^+}(\sqrt{x}+2)=4,$$

  $$\lim_{x\to 4^-} f(x)=\lim_{x\to 4^-}16-3x=16-3\cdot 4=4.$$

  So $\lim\limits_{x\to 4} f(x)=4$. Since $\lim\limits_{x\to 4} f(x)=f(4)$, f is continuous at $x = 4$.

- At $x = -4$, $f(-4) = 16 - 3 \cdot (-4) = 28$.

  Let us find the limit of the function at $x = -4$:

  $$\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \frac{16}{x-4} = -2,$$

  $$\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} 16 - 3x = 16 - 3 \cdot (-4) = 28.$$

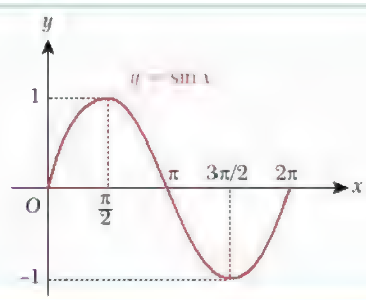
  Since $\lim_{x \to -4^-} f(x) \neq \lim_{x \to -4^+} f(x)$, $\lim_{x \to -4} f(x)$ does not exist.

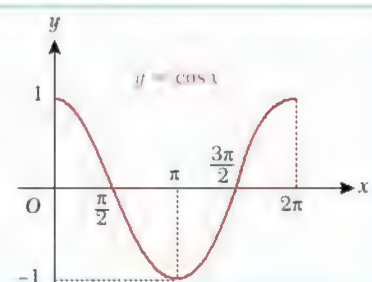
  So f is discontinuous at $x = -4$.

In conclusion, f is continuous on the interval $\mathbb{R} - \{-4\}$.

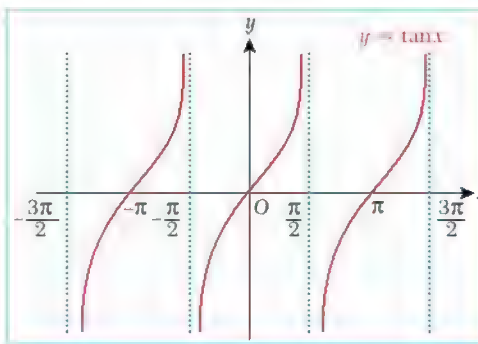
Remark

The trigonometric functions $f(x) = \sin x$ and $f(x) = \cos x$ are continuous for all $x \in \mathbb{R}$.





However, since the function $f(x) = \tan x$ is not defined at the set of points $\frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$, $f(x) = \tan x$ is discontinuous at these points and continuous in the set $\mathbb{R} - \{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$.



Check Yourself 2

1. Find the points, if any, at which each function is discontinuous.

   a. $f(x) = 2x^3 + x^2 - x + 7$ b. $f(x) = \dfrac{x^2 + x - 5}{x + 5}$ c. $f(x) = \left(\dfrac{x^3 - 5x^2 + 1}{x - 7}\right)^{12}$

2. Find the points at which each function is discontinuous.

   a. $f(x) = \dfrac{2}{x - 2}$ b. $f(x) = \dfrac{x}{(x+1)^2}$ c. $f(x) = \dfrac{x + 2}{x^2 - 4x + 3}$

   d. $f(x) = |x - 5|$ e. $f(x) = \dfrac{2}{1 + x^2}$ f. $f(x) = \dfrac{\cos x}{x}$

   g. $f(x) = \cot x$ h. $f(x) = \log(x - 4)$ i. $f(x) = \cos(\ln x)$

   j. $f(x) = \sin\left(\dfrac{4 - x}{x^2 - 1}\right)$

3. Find the largest interval on which each function is continuous.

   a. $f(x) = \sqrt{2x}$ b. $f(x) = \sqrt[3]{x + 1}$ c. $f(x) = \sqrt{x^2 - 2x - 3}$

   d. $f(x) = x - \sqrt{\dfrac{x + 1}{x - 1}}$ e. $f(x) = \log_2(x - 1)$ f. $f(x) = \log_5\left(\dfrac{x + 2}{x - 2}\right)$

Answers

3. a. $[0, \infty)$ b. $\mathbb{R}$ c. $\mathbb{R} - (-1, 3)$ d. $\mathbb{R} - (-1, 1]$ e. $(1, \infty)$ f. $\mathbb{R} - [-2, 2]$

C. TYPES OF DISCONTINUITY

If a function does not satisfy at least one of the three conditions necessary for continuity in its domain then it is called a discontinuous function. In other words, if the graph of a function has a gap, a hole, a break or a jump of some kind in its domain then it is called a discontinuous function.

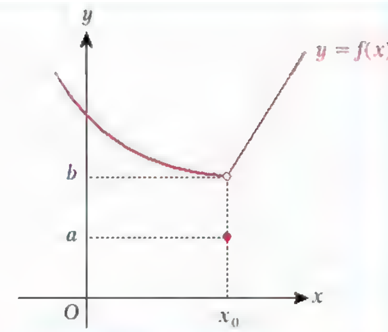
There are three different types of discontinuity:

1. removable discontinuity
2. jump discontinuity
3. infinite discontinuity.

Let us look at each type in turn.

1. Removable Discontinuity

A function f has removable discontinuity at a point x_0 if the limit of the function f at that point exists but is not equal to $f(x_0)$.

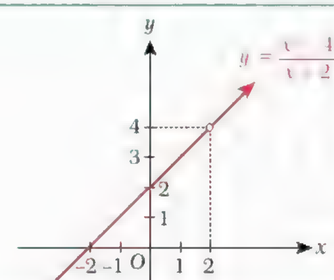


removable discontinuity
$f(x_0) = a$ but $\lim f(x) = b$

We call this type of discontinuity 'removable' because we can remove it by redefining the function. We simply rewrite the function at the point x_0 so that $f(x_0)$ has the same value as $\lim\limits_{x \to x_0} f(x)$.

EXAMPLE 10 Examine the discontinuity of the function $f: \mathbb{R} - \{2\} \to \mathbb{R},\ f(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 2$, and if possible, remove the discontinuity.

Solution The function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is undefined for the point $x = 2$, i.e. $f(2)$ does not exist.



So let us find the limit of f at $x = 2$:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2}$$

$$= \lim_{x \to 2}(x+2) = 2 + 2 = 4.$$

So the limit exists at $x = 2$, but $f(2)$ is not defined. So f has removable discontinuity at the point $x = 2$.

If we redefine the function by specifying $f(2) = 4$, the function will be continuous in the set of real numbers.

$$\text{Hence, } f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2. \end{cases}$$

EXAMPLE 11 Examine the discontinuity of the function $f(x)=\begin{cases} 2x-1 & \text{if } x<1 \\ -1 & \text{if } x=1 \\ 2-x & \text{if } x>1 \end{cases}$ at the point $x=1$, and remove it if possible.

Solution Let us begin by considering the limit at $x = 1$:

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(2x-1)=1$$

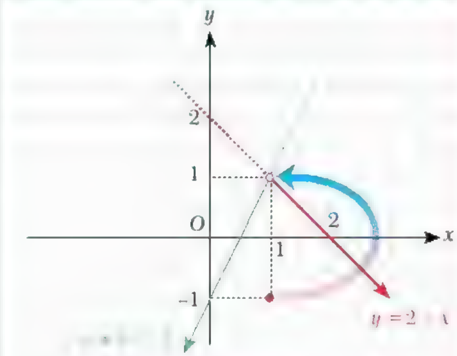
$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(2-x)=1, \text{ so } \lim_{x\to 1} f(x)=1.$$

But $f(1) = -1$ which is not equal to the limit of f at $x = 1$.

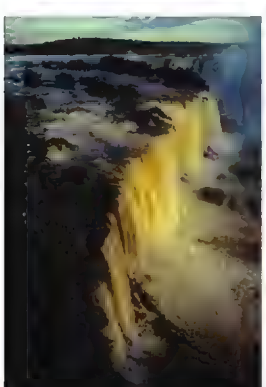
So the function f has removable discontinuity at this point.

In order to remove the discontinuity of f, we must make the value of $f(1)$ the same as the value of the limit at that point:

$$f(x)=\begin{cases} 2x-1 & \text{if } x<1 \\ 1 & \text{if } x=1 \\ 2-x & \text{if } x>1. \end{cases}$$



$f(1)$ should be redefined as $f(1) = 1$

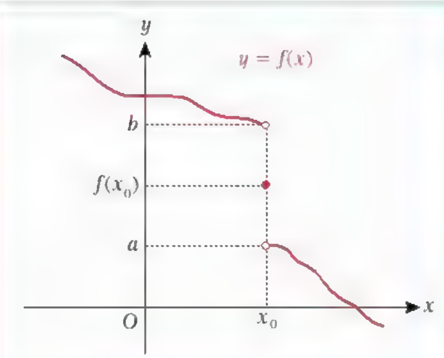


2. Jump Discontinuity

If the right-hand and left-hand limits of the function f are not equal at a point x_0 then f is said to have jump discontinuity at the point x_0.

The figure shows an example of jump discontinuity. As we can see,

$$\lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0^+} f(x).$$



jump discontinuity: $f(x)$ has different right-hand and left-hand limits at x_0

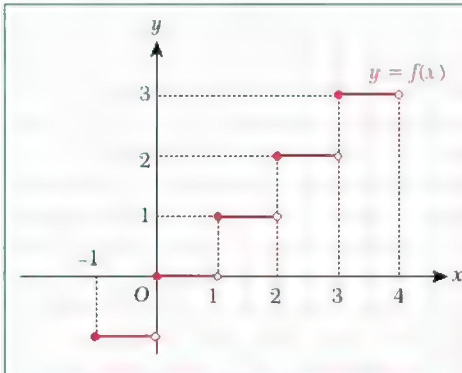
EXAMPLE **12** Examine the discontinuity of the function f defined by $f(x) = [\![x]\!]$.

Solution The function $f(x) = [\![x]\!]$ has jump discontinuity at each integer value. For example:

for $x = 2$, $\lim\limits_{x\to 2^+}[\![x]\!] = 2$ but $\lim\limits_{x\to 2^-}[\![x]\!] = 1$, and

for $x = 1$, $\lim\limits_{x\to 1^+}[\![x]\!] = 1$ but $\lim\limits_{x\to 1^-}[\![x]\!] = 0$.

We can also see this in the graph of $f(x) = [\![x]\!]$. So the function has jump discontinuity.



the graph jumps at each integer value

EXAMPLE **13** $f(x) = \begin{cases} 3x-4 & \text{if} \quad x<1 \\ -x^2+2x+3 & \text{if} \quad x \geq 1 \end{cases}$ is given. Find the point at which f has jump discontinuity and calculate the distance of the jump.

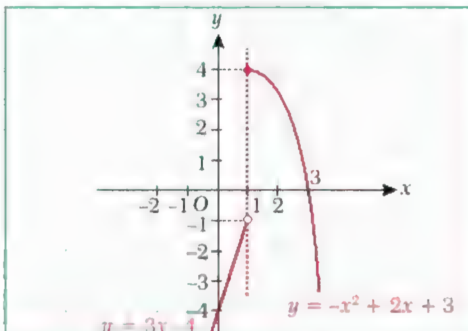
Solution First let us draw the graph of $y = f(x)$.

We can see from the graph that $f(x)$ has jump discontinuity at $x = 1$:

$\lim\limits_{x\to 1^-} f(x) = -1$ and $\lim\limits_{x\to 1^+} f(x) = 4$.

The distance of the jump at this point is

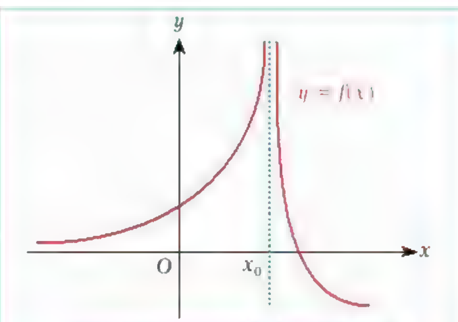
$\lim\limits_{x\to 1^+} f(x) - \lim\limits_{x\to 1^-} f(x) = 4 - (-1) = 5$ units.



3. Infinite Discontinuity

If at least one of the right-hand and left-hand limits of a function f at a point x_0 is infinite then f is said to have infinite discontinuity at the point x_0.

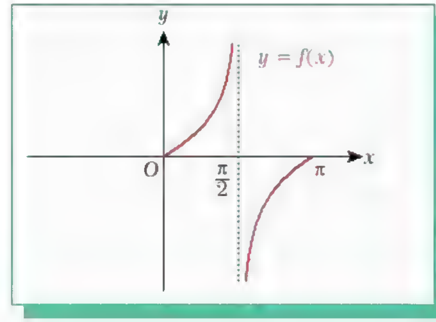
$\lim\limits_{x\to x_0} f(x) = \infty$ means that $f(x)$ has infinite discontinuity at x_0.



infinite discontinuity $\lim\limits_{x\to x_0} f(x) = \infty$

EXAMPLE 14 Examine the discontinuity of the function $f: [0, \pi] \to \mathbb{R}$, $f(x) = \tan x$ at the point $x = \frac{\pi}{2}$.

Solution Let us examine the one-sided limits:



$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$$

but $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \tan x = +\infty$.

Since at least one limit is infinite, the function f has infinite discontinuity at this point.

Check Yourself 3

For each function, find the points of discontinuity. Specify the type of discontinuity, and if possible, remove it.

1. $f(x) = \dfrac{x}{(x+2)^2}$
2. $f(x) = \dfrac{2x}{x^2 - 1}$
3. $f(x) = \dfrac{x-3}{x^2 - 9}$
4. $f(x) = \dfrac{x+1}{x^2 - x - 6}$
5. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } x > 0 \end{cases}$
6. $f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ 3-x & \text{if } x > 1 \end{cases}$
7. $f(x) = \begin{cases} \sin x + 1 & \text{if } x \geq \frac{\pi}{2} \\ \cos x & \text{if } x < \frac{\pi}{2} \end{cases}$
8. $f(x) = [\![x - 1]\!]$
9. $f(x) = \begin{cases} x^2 + 1 & \text{if } x > 0 \\ 2 & \text{if } x = 0 \\ x + 3 & \text{if } x < 0 \end{cases}$
10. $f(x) = \cot x$

Answers

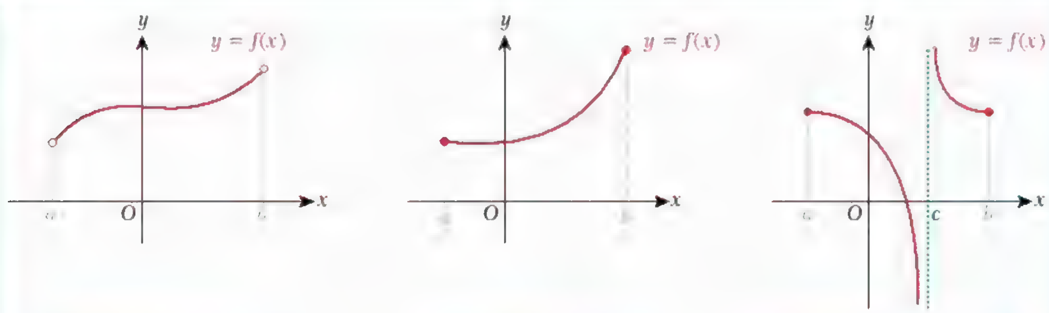
1. infinite discontinuity at $x = -2$
2. infinite discontinuity at $x = -1$ and $x = 1$
3. removable discontinuity at $x = 3$. Remove it by defining $f(x) = \dfrac{1}{6}$ at $x = 3$.
4. infinite discontinuity at $x = -2$ and $x = 3$
5. removable discontinuity at $x = 0$. Remove it by defining $f(x) = 0$ at $x = 0$.
6. removable discontinuity at $x = 1$. Remove it by defining $f(x) = 2$ at $x = 1$.
7. jump discontinuity at $x = \dfrac{\pi}{2}$
8. jump discontinuity at integer values
9. jump discontinuity at $x = 0$
10. infinite discontinuity at $x = k\pi$ ($k \in \mathbb{Z}$)

D. PROPERTIES OF A CONTINUOUS FUNCTION ON A CLOSED INTERVAL

Definition

continuity on an open or closed interval

- If a function f is continuous at each $x \in (a, b)$, it is said to be continuous on the open interval (a, b).
- If a function f is continuous on an interval (a, b) and at the same time $\lim\limits_{x \to a^+} f(x) = f(a)$ and $\lim\limits_{x \to b^-} f(x) = f(b)$, the function is said to be continuous on the closed interval $[a, b]$.



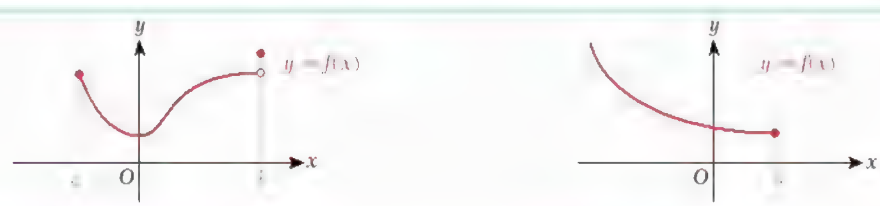
f is continuous on (a, b) f is continuous on $[a, b]$ f is not continuous on $[a, b]$

Note

The continuity of f on the closed interval $[a, b]$ does not mean that f is necessarily continuous at the endpoints $x = a$ or $x = b$.

Remark

Just as we have one-sided limits, we can also define the one-sided continuity of a function on a half-open interval such as $(a, b]$ or on an infinite interval such as $[a, \infty)$. If the one-sided limit of a function is equal to the function value at an endpoint of the interval then the function is called either left-hand side continuous or right-hand side continuous.



f is continuous on $[a, b)$ but not on $[a, b]$
At point a, f is right-hand side continuous.

f is continuous on $(-\infty, a]$
At point a, f is left-hand side continuous.

EXTREME VALUE THEOREM

Let f be a function defined from $[a, b]$ into $\mathbb{R}$. If f is continuous on $[a, b]$ then it has a maximum and a minimum on the interval $[a, b]$.

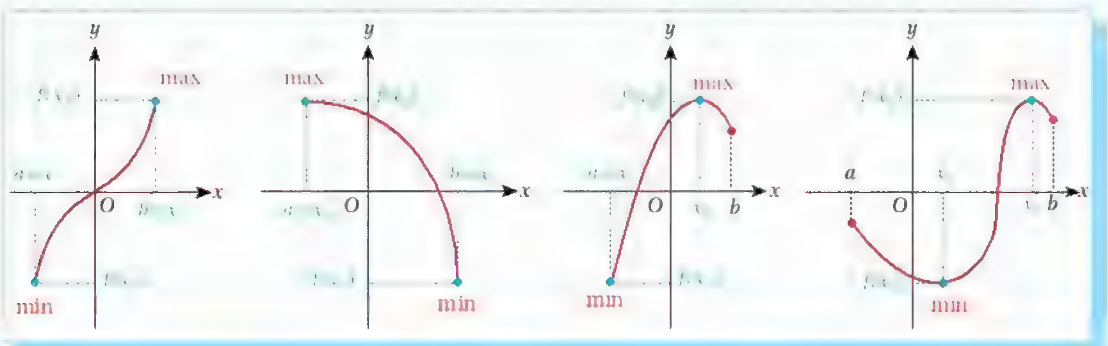
In other words, if f is continuous then there exist two real numbers m and M such that

$$f([a, b]) = [m, M].$$

Equivalently, it is possible to find $x_1, x_2 \in [a, b]$ such that for any $x \in [a, b]$,

$$f(x_1) \le f(x) \le f(x_2).$$

In this case $f(x_1)$ is called the minimum value and $f(x_2)$ is called the maximum value of the function f in the interval $[a, b]$.



EXAMPLE 15 The function $f(x) = x^2 + 3x - 4$ is defined on the closed interval $[-5, 1]$. Find the maximum and minimum values of f on this interval.

Solution $f(x) = x^2 + 3x - 4$ is a polynomial function, so it is continuous on the interval $[-5, 1]$.

At the endpoints, the function takes the values $f(-5) = (-5)^2 + 3(-5) - 4 = 6$ and $f(1) = 1^2 + 3 \cdot 1 - 4 = 0$.

In order to determine whether these values are the maximum and minimum values of f on $[-5, 1]$ or not, let us draw the graph of f.

Calculate the x-intercepts: for $x^2 + 3x - 4 = 0$, the roots are $x_1 = -4$ and $x_2 = 1$.

Calculate the y-intercept: at $x = 0$, $f(0) = -4$.

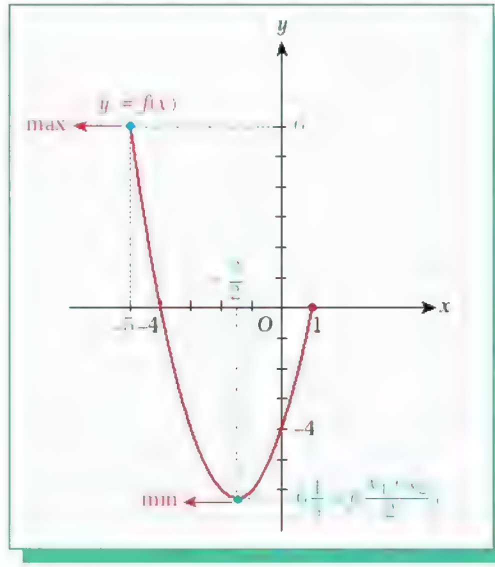
As we can see in the graph, f reaches its maximum value when $x = -5$, so $f(-5) = 6$ is the maximum value of f.

However, $f(1) = 0$ is not the minimum value on this interval. Since f is a polynomial function, it reaches its minimum value at the vertex point, whose abscissa is

$$\frac{x_1 + x_2}{2} = \frac{-4+1}{2} = -\frac{3}{2}.$$

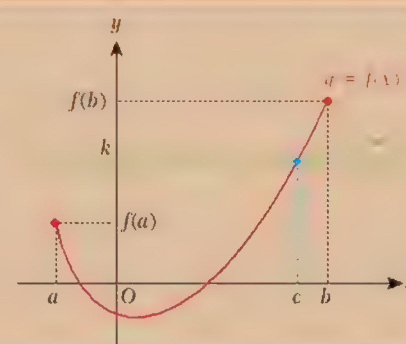
So the minimum value of f is

$$f(-\frac{3}{2}) = (-\frac{3}{2})^2 + 3(-\frac{3}{2}) - 4 = -\frac{25}{4} = -6\frac{1}{4}.$$



INTERMEDIATE VALUE THEOREM

Let f be a function which is continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. Let k be any number between $f(a)$ and $f(b)$. Then there exists at least one real number $c \in (a, b)$ such that $f(c) = k$.



According to the Intermediate Value Theorem, a continuous function cannot 'skip' any value while passing through one value to another.

The figure above shows the graph of a function f. As we can see in the figure, $f(a) \neq f(b)$ and the number k is between $f(a)$ and $f(b)$. At the point $x = c$ the function f intercepts the line $y = k$ such that $f(c) = k$.

In addition, $f(a)$ and $f(b)$ are on opposite sides of the line $y = k$.

As another example, consider the function $f(x) = \sqrt[3]{x^2 + 2}$, which is continuous on the closed interval $[0, 5]$.

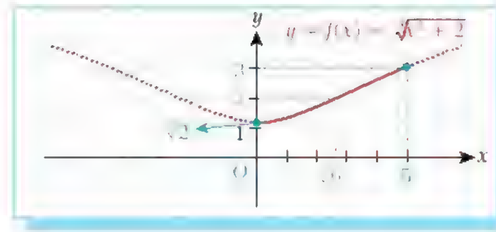
When $x = 0$, $f(0) = \sqrt[3]{2}$ and when $x = 5$, $f(5) = 3$.

By the Intermediate Value Theorem, we can say that for every $k \in [\sqrt[3]{2}, 3]$ there exists at least one $c \in [0, 5]$ such that $f(c) = k$.

Let us choose $k = 2$, then

$f(c) = \sqrt[3]{c^2+2} = 2$, $c^2 + 2 = 8$, $c^2 = 6$

$c = \pm\sqrt{6}$.



Since $c \in [0, 5]$, $c = \sqrt{6}$. This supports the Intermediate Value Theorem.

EXAMPLE 16 Show that $x^2 - 5 = 0$ has a solution on the interval $[2, 3]$.

Solution $f(x) = x^2 - 5$ is a polynomial function and is continuous on $[2, 3]$.

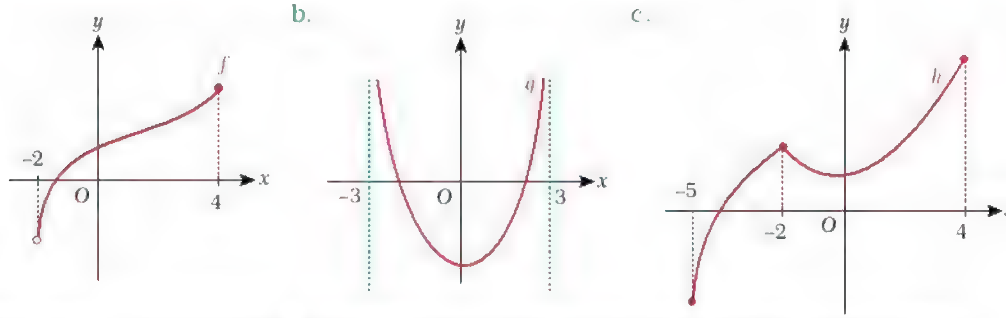
At the endpoints, $f(2) = -1$ and $f(3) = 4$. Since $-1 < k = 0 < 4$, by the Intermediate Value Theorem there exists a real number c between 2 and 3 such that $f(c) = 0$.

Hence, c is a solution of $x^2 - 5 = 0$ on $[2, 3]$.

Check Yourself 4

1. Determine the interval on which each function is continuous.

a. b. c.



2. Given $f(x) = -x^2 + 4$, find the maximum and minimum value of f on each interval.

a. $[-5, 2]$ b. $[1, 4]$

3. $f(x) = \sqrt[3]{x^3+1}$ is defined on the interval $[-3, 5]$. Find a real number c such that

a. $f(c) = -2$. b. $f(c) = 4$.

4. Show that $x^3 + x + 1 = 0$ has a solution on the interval $[-1, 0]$.

Answers

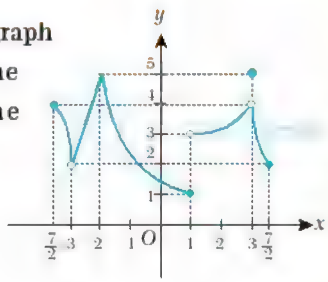
2. a. min = -21, max = 4 b. min = -12, max = 3 3. a. $\sqrt[3]{-9}$ b. $\sqrt[3]{63}$

4. Hint: Use the Intermediate Value Theorem.

EXERCISES 3.1

A. Continuity at a Point

1. The figure shows the graph of a function f. Examine the continuity of f at the integer points in the domain.



2. Examine the continuity of $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x+4 & \text{if } x = 4 \\ 2x & \text{if } x \neq 4 \end{cases} \text{ at the point } x = 4.$$

3. Examine the continuity of $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x+6 & \text{if } x > 3 \\ x^2 & \text{if } x < 3 \end{cases} \text{ at the point } x = 3.$$

4. Find the points at which the function

$$f(x) = \begin{cases} 1 & \text{if } x < 1 \\ 2 & \text{if } 1 \le x \le 2 \\ 1 & \text{if } x > 2 \end{cases} \text{ is discontinuous.}$$

5. Examine the continuity of $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x^2+1 & \text{if } x < 1 \\ x+2 & \text{if } x = 1 \\ -x+4 & \text{if } x > 1 \end{cases} \text{ at the point } x=1.$$

6. The function $f(x) = \begin{cases} x^2+2x+2 & \text{if } x < -1 \\ m & \text{if } x = -1 \\ mx+n & \text{if } x > -1 \end{cases}$

is continuous at $x = -1$. Find $m + n$.

7. The function $f(x) = \begin{cases} mx+3 & \text{if } x > 2 \\ x+1 & \text{if } x = 2 \\ n^2-1 & \text{if } x < 2 \end{cases}$

is continuous at $x = 2$. Find the possible values of m and n.

8. $f(x) = \begin{cases} ax^2 - a^2x + 3 & \text{if } x < -1 \\ a^3x^2 + ax + 5 & \text{if } x \ge -1 \end{cases}$ is given.

Find the possible values of a if f is continuous at the point $x = -1$.

B. Continuity on an Interval

9. Find the largest interval on which each function is continuous.

a. $f(x) = x^3 - x^2 + x - 1$

b. $f(x) = \dfrac{x}{x^2+2}$

c. $f(x) = \dfrac{1}{x-1}$

d. $f(x) = \dfrac{x+2}{x^2-5x-6}$

e. $f(x) = x^3 + \dfrac{1}{x}$

f. $f(x) = \sqrt[3]{2x-5}$

g. $f(x) = \sqrt[3]{3x+4} + \sqrt{x^2-4}$

h. $f(x) = \sqrt{\dfrac{x^2-2x+1}{x^2-3x-4}}$

i. $f(x) = \log_3(x^2-4)$

j. $f(x) = \sqrt{\log_{\frac{1}{2}} \dfrac{x+2}{x}}$

10. Find the largest interval on which each function is continuous.

a. $f(x)=\dfrac{1}{2}\cos x$ b. $f(x)=\dfrac{\sin x}{1-\cos x}$

c. $f(x)=\sqrt{\sin x}$ d. $f(x)=\sin(\dfrac{x+2}{x-2})$

e. $f(x) = \arccos(x-4)$ f. $f(x)=\sin(\log\dfrac{x+2}{x-2})$

g. $f(x)=\arcsin(\dfrac{3x+4}{x-1})$

11. $f(x)=\begin{cases} \dfrac{x^3-27}{3-x} & \text{if } x>3 \\ a & \text{if } x=3 \\ x^2-bx+27 & \text{if } x<3 \end{cases}$

is a continuous function in the set of real numbers. Find $a+b$.

12. $f(x)=\begin{cases} \dfrac{ax}{x^2+1} & \text{if } x>1 \\ ax-2 & \text{if } x\le 1 \end{cases}$

is continuous in $\mathbb{R}$. Find a.

C. Types of Discontinuity

13. For each function, find the point of discontinuity. Specify the type of discontinuity and if possible, remove it.

a. $f(x)=\dfrac{x^2-9}{x-3}$ b. $f(x)=\dfrac{x^3}{x}$

c. $f(x)=\dfrac{x^3+1}{x+1}$ d. $f(x)=\dfrac{|x|}{x}$

14. For each function, find the point(s) of discontinuity. Specify the type of discontinuity and if possible, remove it.

a. $f(x) = \operatorname{sgn} x$ b. $f(x) = [\![x+1]\!]$

c. $f(x)=\begin{cases} x+2 & \text{if } x>1 \\ 2 & \text{if } x=1 \\ 2x+1 & \text{if } x<1 \end{cases}$

d. $f(x)=\begin{cases} 2x & \text{if } x\ge 2 \\ 2 & \text{if } x<2 \end{cases}$

e. $f(x)=\begin{cases} \sin x+1 & \text{if } x\ge \dfrac{\pi}{2} \\ \cos x & \text{if } x<\dfrac{\pi}{2} \end{cases}$

f. $f(x)=\begin{cases} x^2+1 & \text{if } x>0 \\ 2 & \text{if } x=0 \\ x+3 & \text{if } x<0 \end{cases}$

D. Properties of a Continuous Function on a Closed Interval

15. Find the maximum and minimum values of $f(x) = 3x+2$ on the interval $[-3, 3]$.

16. Find the maximum value M and the minimum value m of each function on the given interval.

a. $f(x) = 1-x$ on $[-1, 1]$

b. $f(x) = |x|$ on $[-2, 3]$

c. $f(x) = \dfrac{1}{\sqrt{x}}$ on $[1, 2]$

d. $f(x) = 5-x^2$ on $[-1, 2]$

e. $f(x) = x^3+1$ on $[-1, 3]$

17. $f: [0, 4] \to \mathbb{R}$, $f(x) = x^2 - 2$ is given.

a. Use the Extreme Value Theorem to find the maximum and minimum values of f.

b. By applying the Intermediate Value Theorem, show that $f(x)$ takes the values -1, 0 and 11.

18. Find the image set of the function $f(x) = 5 - x^2$ in the interval $[-\frac{7}{2}, 3]$ and determine its maximum and minimum values in this interval.

19. $f(x)=\begin{cases} -x-5 & \text{if } x<-2 \\ x^2+3x-1 & \text{if } x\geq -2 \end{cases}$ is given.

a. Find $f([-4, 4])$.

b. Find the maximum value M and minimum value m of f on $[-4, 4]$.

20. Show that each equation has a solution on the specified interval.

a. $x^3 - 3x^2 + 1 = 0$ on $[0, 1]$

b. $x^3 - 5 = 0$ on $[1, 2]$

c. $x^4 + 2x - 1 = 0$ on $[0, 1]$

d. $x^5 - 5x^3 + 3 = 0$ on $[-3, -2]$

21. Show that there is a number x between 1 and 2 such that $x^3 - 1 = x$.

22. For which positive values of x does the value of $f(x)=\dfrac{x}{x+1}$ lie between $\dfrac{1}{4}$ and $\dfrac{3}{4}$?

Mixed Problems

23. Given $f(x)=\dfrac{x^2-2}{4x^2+mx+4}$, determine the interval of m which makes f continuous for all real numbers.

24. Determine the points at which the function $f(x)=\dfrac{1}{[\![x+2]\!]}$ is not continuous.

25. $f(x) = \operatorname{sgn}(ax^2 + bx + c)$ is continuous except at the points $x = 2$ and $x = 3$. Find $\dfrac{b}{a}$.

26. $f(x)=\begin{cases} x^2+1 & \text{if } x<0 \\ 3x-8 & \text{if } x\geq 0 \end{cases}$ is given. Find the point at which f has jump discontinuity and calculate the distance of the jump.

27. $f(x)=\begin{cases} x^2+x+4 & \text{if } x\leq 1 \\ x^3+2 & \text{if } x>1 \end{cases}$ is given. Find the point at which f has jump discontinuity and calculate the distance of the jump.

28. Study the discontinuity of each function and determine its type.

a. $f(x) = \sec x$

b. $f(x)=\dfrac{x}{\log x}$

c. $f(x)=\dfrac{x}{\operatorname{sgn} x}$

d. $f(x)=\dfrac{x^2+1}{x^2-1}$

29. Find all values of a which make the function

$$f(x)=\begin{cases} a^2x-a & \text{if } x\geq 2 \\ 3 & \text{if } x<2 \end{cases}$$

continuous for all real values of x.

30. Find the values of a and b which make the function

$$f(x)=\begin{cases} ax+b & \text{if } x\leq -2 \\ 2x^2+3ax+b & \text{if } -2<x\leq 2 \\ 5 & \text{if } x>2 \end{cases}$$

continuous for all real values of x.

CHAPTER SUMMARY

- A function f is continuous at a point x_0 if
  1. the limit of $f(x)$ as $x \to x_0$ exists,
  2. the function f is defined at the number x_0, i.e. $f(x_0)$ exists, and
  3. $\lim_{x \to x_0} f(x) = f(x_0)$.

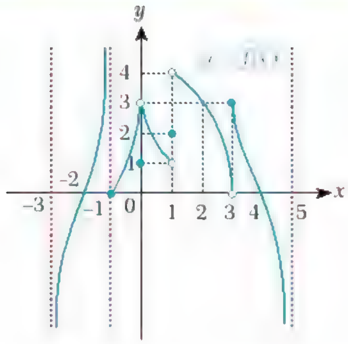
  If even one of these three conditions is not satisfied, f is said to be discontinuous at the point x_0.
- If f and g are two continuous functions at $x = a$ then the following functions are also continuous at $x = a$:

  1. $f + g$ 2. $f - g$ 3. $c \cdot f\ (c \in \mathbb{R})$ 4. $f \cdot g$

  5. $\frac{f}{g}$, $(g(a) \neq 0)$
- If $f: A \to \mathbb{R}$, $y = f(x)$ is continuous for all $x \in A$ then it is said to be a continuous function in its domain A.
- Continuity of common types of function:
  1. A polynomial function $P(x)$ is defined for all $x \in \mathbb{R}$ and $\lim_{x \to x_0} P(x) = P(x_0)$. So P is continuous at every point in the interval $(-\infty, +\infty)$.
  2. A rational expression $f(x) = \frac{P(x)}{Q(x)}$ is continuous at all points in the interval $(-\infty, +\infty)$ such that $Q(x) \neq 0$.
  3. For $n \in \mathbb{Z}^+$, the function $f(x) = \sqrt[2n]{P(x)}$ is defined when $P(x) \geq 0$, and $g(x) = \sqrt[2n+1]{P(x)}$ is defined for all $x \in \mathbb{R}$.
  4. For $a \in \mathbb{R}^+ - \{1\}$, the logarithmic function $f(x) = \log_a P(x)$ is continuous for $x \in \mathbb{R}$ satisfying $P(x) > 0$.
  5. For $a \in \mathbb{R}^+ - \{1\}$, the exponential function $f(x) = a^{P(x)}$ is continuous for $x \in \mathbb{R}$.
- If a function f is discontinuous at a given point x_0 then it can have one of three types of discontinuity:
  1. removable discontinuity: $f(x_0) = a$ but $\lim_{x \to x_0} f(x) \neq a$.
  2. jump discontinuity: $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$.
  3. infinite discontinuity: $\lim_{x \to x_0} f(x) = \pm\infty$.
- If a function f is continuous at each $x \in (a, b)$, f is said to be continuous on the open interval (a, b).
- If a function f is continuous on (a, b) and at the same time $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$, f is said to be continuous on the closed interval $[a, b]$.
- If the one-sided limit of a function is equal to the function value at an endpoint of the interval then the function is called either left-hand side continuous or right-hand side continuous.
- Extreme Value Theorem: Let f be a function defined from $[a, b]$ into $\mathbb{R}$. If f is continuous on $[a, b]$ then it has a maximum and a minimum on the interval $[a, b]$.
- Intermediate Value Theorem: Let f be a function which is continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. Let k be any number between $f(a)$ and $f(b)$. Then there exists at least one real number $c \in (a, b)$ such that $f(c) = k$.

Concept Check

1. What are the three conditions which guarantee the continuity of a function at a point?
2. What is the largest interval in which a polynomial function can be continuous?
3. What is the largest interval in which a rational function can be continuous?
4. What is the largest interval in which a radical function with an odd index can be continuous?
5. What is the largest interval in which a radical function with an even index can be continuous?
6. What is the largest interval in which an exponential function can be continuous?
7. What is the largest interval in which a logarithmic function can be continuous?
8. What is the largest interval in which the function $y = \sin x$ is continuous?
9. What is the largest interval in which the function $y = \cot x$ is continuous?
10. In which situation can the discontinuity of a function at a point be removed?
11. What is jump discontinuity?
12. When does infinite discontinuity occur?
13. If a function is continuous on a closed interval $[a, b]$, what does this mean?
14. Explain one-sided continuity.
15. Explain the concepts of minimum and maximum value of a function.
16. State the Extreme Value Theorem.
17. State the Intermediate Value Theorem.
18. How can we determine whether or not an equation has a solution in a given closed interval?

1. 

The graph of a function f is shown in the figure. Find the number of integer points in the interval $(-3, 5)$ at which the function is continuous.

A) 1 B) 2 C) 3 D) 4 E) 5

2. Find the points at which the function $f(x) = \text{sgn}(x^2 - 6x + 5)$ is discontinuous.

A) $x \in \{-1, -5\}$ B) $x \in \{1, 5\}$ C) $x \in \{1\}$ D) $x \in \{-5\}$ E) $x \in \{-5, 1\}$

3. Find the largest interval on which the function $f(x) = |x - 2|$ is continuous.

A) $x \in (-\infty, 2)$ B) $x \in [2, \infty)$ C) $x \in (-2, 2)$ D) $x \in \mathbb{R}^+$ E) $x \in \mathbb{R}$

4. For what values of x is the function $f(x) = \sqrt{x^2 - 5x}$ discontinuous?

A) $x \in (0, 5)$ B) $x \in [0, 5]$ C) $x \in (-\infty, 0)$ D) $x \in (5, \infty)$ E) $x \in (0, 5]$

5. The function $f(x) = \begin{cases} ax + b & \text{if } x > -2 \\ 5 & \text{if } x = -2 \\ -b & \text{if } x < -2 \end{cases}$ is continuous at $x = -2$. Find $a + b$.

A) 5 B) 10 C) -10 D) -5 E) 0

6. At which point is the function $f(x) = \dfrac{|x-3| + |3-x|}{|9-3x|}$ discontinuous?

A) $x = 3$ B) $x = -3$ C) $x = 0$ D) $x = 9$ E) $x = -9$

7. What is the largest interval on which $f(x) = \sqrt{\log_2 x} + \sqrt{\log_x 2}$ is continuous?

A) $x \geq 1$ B) $x > 1$ C) $x < 1$ D) $x \leq 1$ E) $x = 1$

8. The function $f(x) = \dfrac{x^2 + 6}{x^2 + x - a}$ is discontinuous at $x = 2$. Find a.

A) -3 B) 0 C) 2 D) 3 E) 6

9. $f(x)=\begin{cases} x^2+2 & \text{if } x \le -1 \\ 3x-5 & \text{if } x > -1 \end{cases}$ has jump discontinuity.

Calculate the distance of the jump.

A) 7 B) 8 C) 9 D) 10 E) 11

10. Which value of $f(4)$ will make the function $f(x)=\dfrac{x^2-16}{x^2-3x-4}$ continuous at $x = 4$?

A) $\dfrac{3}{5}$ B) 1 C) $\dfrac{8}{5}$ D) 2 E) $\dfrac{13}{5}$

11. The function $f(x) = \log_4(x^4 - 6x^2 + a)$ is continuous at all $x \in \mathbb{R}$. Find a.

A) $|a| > 3$ B) $a > 5$ C) $a > 9$ D) $|a| > 9$ E) $a > 3$

12. For which positive values of x does $f(x)=\dfrac{2x}{x+1}$ lie between $\dfrac{1}{3}$ and $\dfrac{2}{3}$?

A) $\dfrac{1}{3}<x<\dfrac{2}{3}$ B) $\dfrac{1}{2}<x<\dfrac{2}{3}$ C) $\dfrac{1}{5}<x<\dfrac{1}{2}$ D) $x>\dfrac{1}{5}$ E) $x>\dfrac{1}{2}$

13. What is the minimum value of $f(x) = x^2 - 5x + 6$ in the interval $[0, 3]$?

A) -1 B) $\dfrac{3}{4}$ C) $-\dfrac{3}{4}$ D) $-\dfrac{1}{4}$ E) $\dfrac{1}{4}$

14. What is the maximum value of $f(x)=\sqrt{\log_{\frac{1}{2}} x}$ in the interval $\left[\dfrac{1}{16}, 1\right]$?

A) 1 B) 2 C) 3 D) 4 E) 5

15. At which point does the function $f(x)=\dfrac{x+5}{x+4}$ have infinite discontinuity?

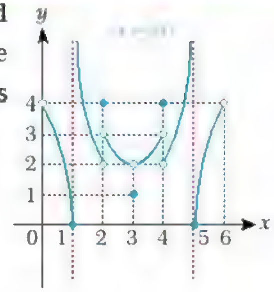
A) $x = -5$ B) $x = -4$ C) $x = 0$ D) $x = 4$ E) $x = 5$

16. Find the largest interval on which the function $f(x) = \arccos(2x + 1)$ is continuous.

A) $[-1, 0]$ B) $(-1, 0)$ C) $[-1, 1]$ D) $(-1, 1)$ E) $[0, 1]$

CHAPTER REVIEW TEST 3B

1. The function f is graphed in the figure. Which of the following statements is <u>false</u>?



A) f is continuous on $(2, 3)$

B) f is left-hand continuous at $x = 1$

C) f is continuous on $(4, \infty)$

D) f is right-hand continuous at $x = 5$

E) f is continuous on $(3, 4)$

2. The function $f(x)=\begin{cases} ax+2 & \text{if } x>3 \\ 2x+2 & \text{if } x=3 \\ b-1 & \text{if } x<3 \end{cases}$

is continuous at $x = 3$. Find $a + b$.

A) 2 B) 4 C) 6 D) 11 E) 15

3. Which of the following functions is continuous in the set of real numbers?

A) $[\![x + 1]\!]$ B) $\operatorname{sgn}(x + 3)$ C) $\cot x$

D) $|x^2 - 4|$ E) $\dfrac{|x|}{x}$

4. The function $f(x)=\begin{cases} 3\cdot 5^x & \text{if } x<1 \\ 3m+4x & \text{if } x\geq 1 \end{cases}$

is continuous at $x = 1$. Find m.

A) $\dfrac{11}{2}$ B) $\dfrac{11}{3}$ C) $\dfrac{11}{4}$ D) $\dfrac{11}{5}$ E) $\dfrac{11}{6}$

5. What is the largest interval on which

$f(x)=\begin{cases} \dfrac{\sin 5x}{x} & \text{if } x\neq 0 \\ 5 & \text{if } x=0 \end{cases}$ is continuous?

A) $(0, 1)$ B) $[-1, 1]$ C) $\mathbb{Z}$ D) $\mathbb{R}^+$ E) $\mathbb{R}$

6. The function $f(x)=\dfrac{x^3-7x+13}{x^2-ax+11}$ is continuous in $\mathbb{R}$. Find the sum of the possible positive integer values of a.

A) 18 B) 21 C) 28 D) 36 E) 49

7. $f(x)=\dfrac{\cos x}{[\![\cos x]\!]}$ is given. What is the largest interval on which f is continuous in the interval $[0, 2\pi)$?

A) $[0, \dfrac{\pi}{2})$ B) $[\dfrac{\pi}{2}, \pi)$ C) $[\dfrac{3\pi}{2}, 2\pi)\cup[0, \dfrac{\pi}{2})$

D) $(\dfrac{\pi}{2}, \dfrac{3\pi}{2})$ E) $[\dfrac{3\pi}{2}, 2\pi)$

8. Given $f(x) = x^2 + 2x + 1$ and $g(x) = 2x^2 - x + 1$, the function h defined as

$h(x)=\begin{cases} (m\cdot f)(x) & \text{if } x>1 \\ x+1 & \text{if } x=1 \\ (n\cdot g)(x) & \text{if } x<1 \end{cases}$

is continuous at $x = 1$. Find $\dfrac{m}{n}$.

A) $\dfrac{1}{2}$ B) 1 C) $\dfrac{3}{2}$ D) 2 E) $\dfrac{5}{2}$

9. For which value(s) of x is the function

$$f:\mathbb{R}\to\mathbb{R},\ f(x)=\begin{cases}\dfrac{x}{x+1} & \text{if} \quad x<-1\\ \dfrac{1}{2} & \text{if} \quad -1\le x\le 1,\\ \dfrac{x}{x+1} & \text{if} \quad x>1\end{cases}$$

discontinuous.

A) $x \in \{1\}$ B) $x \in \{-1\}$ C) $x \in \{0\}$

D) $x \in \{-1, 1\}$ E) $x \in \{-1, 0, 1\}$

10. Find the point at which $f(x) = |\text{sgn}(x - 3)|$ is discontinuous.

A) $x = -3$ B) $x = 0$ C) $x = 3$

D) $x = 6$ E) $x = 9$

11. Find the largest interval on which $f(x)=\sqrt{\ln x}+\sqrt{\log_x e}$ is continuous.

A) $x \le e$ B) $x < e$ C) $x > e$

D) $x > 1$ E) $x > 0$

12. What is the largest interval on which $f(x)=\ln(\dfrac{x-2}{x+1})$ is discontinuous?

A) $[-1, 2]$ B) $(-1, 2)$ C) $[-2, 1]$

D) $(-2, 1)$ E) $[-1, 2)$

13. For which value(s) of x is the function

$$f(x)=\begin{cases}\dfrac{x-4}{\sqrt{x}-2} & \text{if} \quad x>4\\ 3+5x & \text{if} \quad -1\le x\le 4\\ \dfrac{8}{x-3} & \text{if} \quad x<-1\end{cases}$$

discontinuous?

A) $x \in \{-1\}$ B) $x \in \{-1, 4\}$ C) $x \in \{4\}$

D) $x \in \{3, 4\}$ E) $x \in \{-1, 3, 4\}$

14. For which value of k is the function

$$f(x)=\begin{cases}30-x & \text{if } x<3\\ (x-k)^3 & \text{if } x\ge 3\end{cases}$$

continuous for all $x \in \mathbb{R}$?

A) $k = 4$ B) $k = 3$ C) $k = 2$

D) $k = 1$ E) $k = 0$

15. $f(x)=\begin{cases}x^2-1 & \text{if } x<2\\ 5x+1 & \text{if } x\ge 2\end{cases}$ has jump discontinuity.

Calculate the distance of the jump.

A) 3 B) 5 C) 8 D) 9 E) 11

16. Find the value of $f(5)$ which makes

$$f(x)=\begin{cases}x^2-2x & \text{if } x>5\\ \dfrac{7x-5}{2} & \text{if } x<5\end{cases}$$

continuous for all $x \in \mathbb{R}$.

A) 0 B) 5 C) 10 D) 15 E) 20

ANSWERS TO EXERCISES

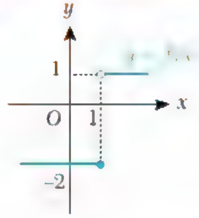
EXERCISES 1.1

1. a. $f: \mathbb{R} \to \mathbb{R}$ b. $g: \mathbb{R} - \{5\} \to \mathbb{R}$ c. $h: \mathbb{R} - \{0\} \to \mathbb{R} - \{3\}$ d. $t: (-4, \infty) - \{1\} \to (-2, 3]$ 2. a. $\mathbb{R} - \{3\}$ b. $\mathbb{R} - [-3, -2]$ c. $(-1, \infty)$ d. $[-\sqrt{13}, \sqrt{13}]$ e. $\mathbb{R} - [-1, 4)$ f. $\mathbb{R} - (0, 5]$ g. $\mathbb{R} - (-2, 5]$ h. $[0, 2\pi] - (\frac{\pi}{6}, \frac{5\pi}{6})$ i. $[\frac{1}{2}, \infty)$ j. $(-\infty, 0]$ k. $(\frac{1-\sqrt{17}}{2}, 0) \cup (\frac{1+\sqrt{17}}{2}, \infty)$ l. $(5, \infty)$ m. $(0, 0.01) \cup (100000, \infty)$ n. $[\frac{\pi}{2}, \frac{3\pi}{2}]$ 3. a. $[-11, 7]$ b. $(-3, 5]$ c. $[5, 8)$ d. $(-4, -2)$ e. $(27, 2187)$ f. $(\frac{1}{16}, 1]$ 4. a. $[0, 2]$ b. $[-1, 3]$ c. $(0, 2)$ d. $[0, 4]$ 5. a. $(\sqrt{x} + 1)^2$ b. $\sqrt{x^2+1}$ 6. a. $g(x) = x + 1$, $h(x) = \sqrt{x}$, $t(x) = \frac{1}{x}$, $r(x) = 5 - x$, $f(x) = r(t(h(g(x))))$ b. $g(x) = x^2 + 5$, $h(x) = \frac{1}{x}$, $t(x) = \log_3 x$, $f(x) = t(h(g(x)))$ 7. a. $f^{-1}: \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = 2 - 5x$ b. $f^{-1}: \mathbb{R} - \{2\} \to \mathbb{R} - \{3\}$, $f^{-1}(x) = \frac{-3x-3}{x+2}$ c. $f^{-1}: \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = \log_3 \sqrt[3]{\frac{x}{162}}$ d. $f^{-1}: \mathbb{R} \to \mathbb{R}^+$, $f^{-1}(x) = \frac{e^{x-2}+4}{5}$ e. $f^{-1}: [1, +\infty) \to [1, +\infty)$, $f^{-1}(x) = \frac{\sqrt{x-1}+2}{2}$

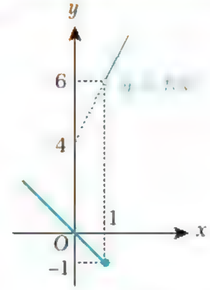
8. a. $\frac{1}{2}$ b. 2 9. $a = \frac{3}{2}$, $b = 4$ 10. a. increasing b. increasing c. decreasing d. increasing 11. a. $(-\infty, \frac{7}{4}]$ b. $[\frac{7}{4}, \infty)$ 12. a. odd b. even c. odd d. odd e. neither even nor odd f. odd

EXERCISES 1.2

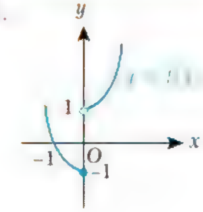
1. 2 2. a.



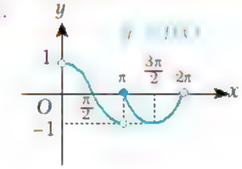
b.



c.



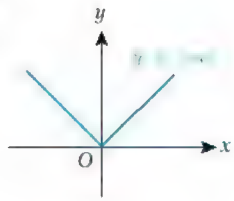
d.



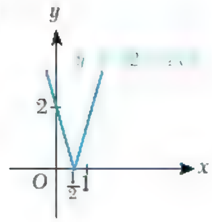
3. a. $f(x) = \begin{cases} x+3 & \text{if } x \geq -3 \\ -x-3 & \text{if } x < -3 \end{cases}$ b. $f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ c. $f(x) = \begin{cases} x^2 - x - 2 & \text{if } x \notin (-1, 2) \\ -x^2 + x + 2 & \text{if } x \in (-1, 2) \end{cases}$

d. $f(x) = \begin{cases} -2x+5 & \text{if } x < 2 \\ 1 & \text{if } 2 \leq x < 3 \\ 2x-5 & \text{if } x > 3 \end{cases}$ e. $f(x) = \begin{cases} -2x & \text{if } x < -1 \\ -2 & \text{if } -1 \leq x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$ f. $f(x) = \begin{cases} 2x^2 + 2x & \text{if } x \geq -4 \\ -6x & \text{if } x < -4 \end{cases}$

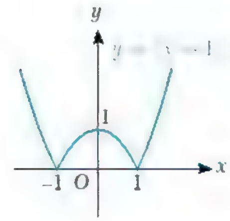
4. a.



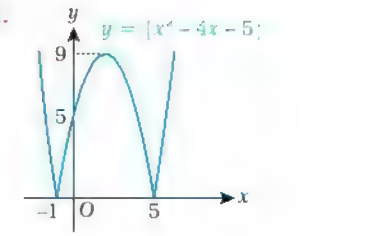
b.



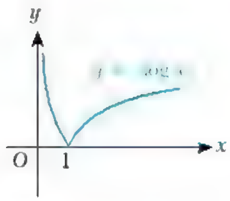
c.



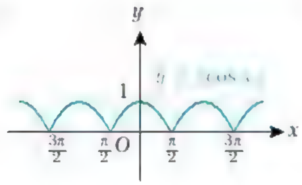
d.



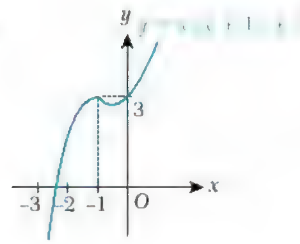
e.



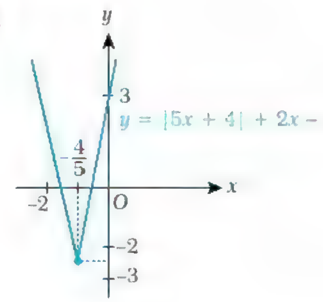
f.



g.



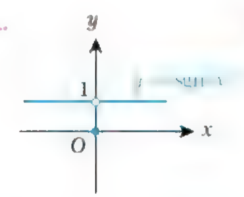
h.



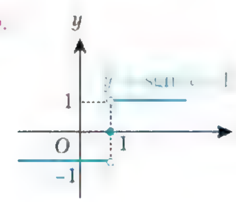
5. a. $x > 0$ b. $x < 1$ c. $x \in (-1, 1)$

d. $x \in \{-1, 0, 1\}$ e. $x \in (1, \infty)$ f. $x = 2$

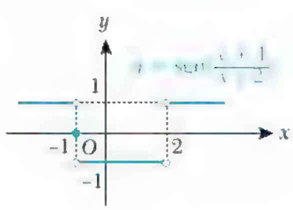
6. a.



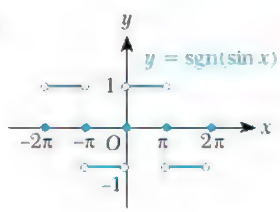
b.



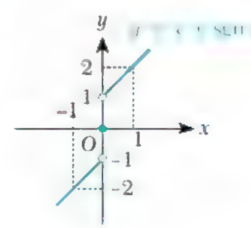
c.



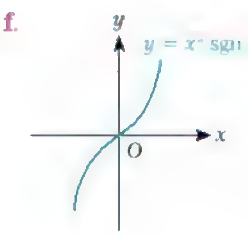
d.



e.

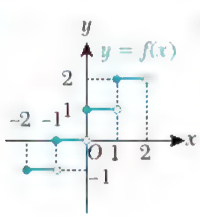


f.

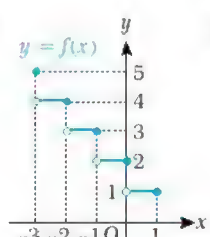


7. a. $x \in [-2, -1)$ b. $x \in [2, 3)$ c. $x \in [-3, -2)$ d. $x \in [1, \frac{5}{2})$ e. $x \in (-\frac{7}{3}, -1]$ f. $x \in [8, 16)$

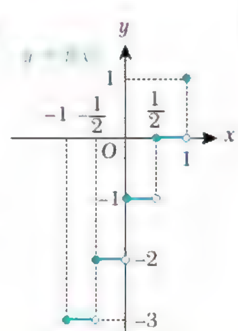
8. a.



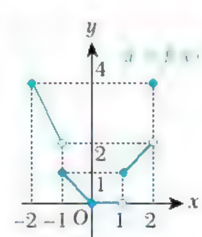
b.



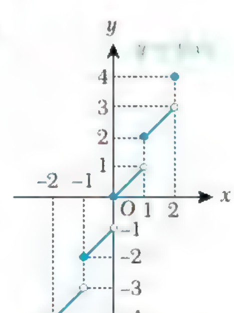
c.



d.



e.



9. a. $x \in \{-1, 4\}$ b. $x \in (-4, -2] \cup \{0, 3\}$ 10. $f(g(x)) = \begin{cases} x^2 + 2x & \text{if } x > 1 \\ 3x + 4 & \text{if } 0 < x \le 1 \\ 3x + 7 & \text{if } x \le 0 \end{cases}$ 11. $f^{-1}: \mathbb{R} \to \mathbb{R}, f^{-1}(x) = \begin{cases} \dfrac{x+1}{4} & \text{if } x > 3 \\ \dfrac{x-1}{2} & \text{if } x \le 3 \end{cases}$

12. a. $f(x) = \begin{cases} x - 1 & \text{if } x > 1 \\ 1 - x & \text{if } x \le 1 \end{cases}$ b. $f(x) = \begin{cases} x^2 - 3x - 4 & \text{if } x \notin (0, 3) \\ -x^2 + 3x - 4 & \text{if } x \in (0, 3) \end{cases}$ c. $f(x) = \begin{cases} x^3 - x & \text{if } x \ge 0 \\ -x^3 - x & \text{if } x < 0 \end{cases}$

d. $f(x) = \begin{cases} \dfrac{3^x . x^2}{x - 4} & \text{if } x > 4 \\ -\dfrac{3^x . x^2}{x - 4} & \text{if } x < 4 \end{cases}$ 13. a. $f(x) = \begin{cases} x^2 & \text{if } x \ge 0 \\ -x^2 & \text{if } x < 0 \end{cases}$ b. $f(x) = \begin{cases} 2x - 1 & \text{if } x \ge 1 \\ 1 & \text{if } x < 1 \end{cases}$

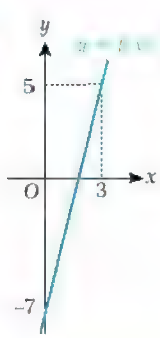
c. $f(x) = \begin{cases} 2x + 3 & \text{if } x \notin (-1, 3) \\ 2x^2 - 2x - 3 & \text{if } x \in (-1, 3) \end{cases}$ d. $f(x) = \begin{cases} -2x + 1 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x < 1 \\ 2x - 1 & \text{if } x \ge 1 \end{cases}$ e. $f(x) = \begin{cases} -1 & \text{if } x < 2 \\ 2x - 5 & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$

f. $f(x) = \begin{cases} x^2 - x - 1 & \text{if } x \le -1 \\ -x^2 - x + 1 & \text{if } -1 < x < 0 \\ -x^2 + x + 1 & \text{if } 0 \le x < 1 \\ x^2 + x - 1 & \text{if } x \ge 1 \end{cases}$ 14. a. $f(x) = \begin{cases} x - 2 & \text{if } x < -2 \\ x - 1 & \text{if } x = -2 \\ x & \text{if } -2 < x < 1 \\ -x + 2 & \text{if } x \ge 1 \end{cases}$ b. $f(x) = \begin{cases} -x & \text{if } x < 0 \text{ or } x > 1 \\ x & \text{if } 0 \le x < 1 \\ 0 & \text{if } x = 1 \end{cases}$

c. $f(x) = \begin{cases} -2x - 1 & \text{if } x < 0 \\ -1 & \text{if } x \ge 0 \end{cases}$ d. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$ 15. $f(x) = x + 1$ 16. 4

EXERCISES 2.1

1. 17 2. 24 3. 27 4. 17 5. $2k + 1$ 6. $2a^3 + a^2$ 7. $b^2 - 2b + 1$

8. 

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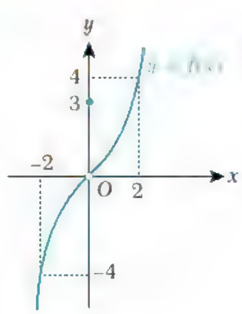
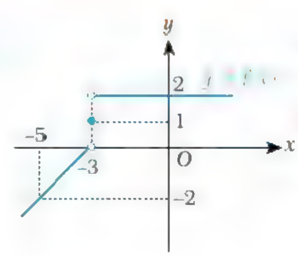
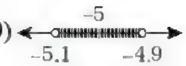
9. 

a. 0 b. 4 c. –4

10. 

a. –2 b. does not exist c. 2

11. (11.7, 12.3) 12. (–5.1, –4.9) 

13. (3.99, 4.01) 14. (4.95, 5.05) 16. a. –3 b. 1 c. –1 d. 0 e. 1 f. 2 17. $p \in \{-2, -1, 0, 1, 3\}$ 18. a. $\lim\limits_{x\to -1^+} f(x) = 0$, $\lim\limits_{x\to -1^-} f(x) = 1$, $\lim\limits_{x\to 0^+} f(x) = 1$, $\lim\limits_{x\to 0^-} f(x) = 1$, $\lim\limits_{x\to 1^+} f(x) = 1$, $\lim\limits_{x\to 1^-} f(x) = 0$ b. $x = -1$, $x = 1$ 19. a. $\lim\limits_{x\to 0} f(x) = -4$, $\lim\limits_{x\to 1} f(x) = -3$ b. $x = -2$ 20. $x = 0$ 21. a. does not exist b. 1 22. 28 23. a. 1 b. 0 c. 1 d. –1 e. –1 f. –1 24. a. 5 b. –1 c. 0 d. $\frac{15}{8}$ e. –11 f. 0 25. a. 4 b. does not exist c. –1 d. –2 e. does not exist f. 0 g. 31 26. a. –2 b. 0 c. ∞ d. does not exist e. 1 27. a. $\frac{2}{3}$ b. 1 c. 1 d. –2 e. –2 f. 2 28. a. ∞ b. ∞ c. $-\infty$ d. does not exist e. $-\infty$ 29. a. 0 b. $-\infty$ c. ∞ d. ∞ e. $-\infty$ f. $-\infty$ g. ∞ h. 0 30. a. $-\infty$ b. $-\infty$ c. $-\infty$ d. $-\infty$ 31. –6 32. a. $\frac{1}{2}$ b. 16 c. 0 d. $2\sqrt{3}$ e. $\frac{\sqrt[3]{6}}{3}$ 33. a. 1 b. 1 c. –1 d. $\frac{1}{2}$ e. 0 f. 2 34. a. 0 b. 2 35. 4 36. –1 37. a. –1 b. ∞ c. –1 d. ∞

EXERCISES 2.2

1. a. $\frac{1}{10}$ b. $\frac{5}{3}$ c. –7 d. 15 e. 6 f. 2 g. 1 h. m i. 16 2. a. $\frac{5}{7}$ b. $\frac{1}{2}$ c. 3 d. $\frac{\sqrt{2}}{2}$ e. $\frac{1}{\sqrt{2}}$ f. $\frac{1}{2}$ g. –1 h. 1 i. 2 j. –6 3. a. 3 b. 0 c. $-\infty$ d. $-\infty$ e. 0 f. 1 g. –1 h. 2 i. 0 j. 4 k. $2\sqrt{3}$ l. $\frac{1}{2}$ 4. a. $\frac{6}{7}$ b. 5 c. 3 d. $\frac{2}{5}$ e. $\frac{1}{3}$ f. $\frac{2}{\pi}$ 5. a. $-\frac{4}{9}$ b. $-\frac{1}{2}$ c. $-\infty$ d. $\frac{1}{2}$ e. 0 f. 0 6. a. $\frac{5}{2}$ b. $-\frac{1}{2}$ c. 2 d. $\frac{1}{2}$ e. 1 f. 0 7. a. e^2 b. e^{-9} c. e^3 d. $e^{1/3}$ e. $e^{-4/3}$ f. $e^{6/5}$ g. $e^{14/5}$ h. e^{-18} i. e^{-1} j. e^4 8. a. 10 b. 0 c. 2 d. 0 e. 0 f. 0 g. $-\frac{1}{2}$ h. 1 9. a. does not exist b. 0

EXERCISES 3.1

1. continuous at $x = -2$, $x = -1$, $x = 0$, $x = 2$, discontinuous at $x = -3$, $x = 1$, $x = 3$ 2. continuous 3. discontinuous 4. $x = 1$, $x = 2$ 5. continuous 6. 3 7. $m = 0$, $n = \pm 2$ 8. $a \in \{1, \pm\sqrt{2}\}$ 9. a. $\mathbb{R}$ b. $\mathbb{R}$ c. $\mathbb{R} - \{1\}$ d. $\mathbb{R} - \{-1, 6\}$ e. $\mathbb{R} - \{0\}$ f. $\mathbb{R}$ g. $\mathbb{R} - (-2, 2)$ h. $\mathbb{R} - [-1, 4]$ i. $\mathbb{R} - [-2, 2]$ j. $(-\infty, -2)$ 10. a. $\mathbb{R}$ b. $\mathbb{R} - \{x \mid x = 2k\pi, k \in \mathbb{Z}\}$ c. $\{x \mid 2k\pi \le x \le (2k + 1)\pi, k \in \mathbb{Z}\}$ d. $\mathbb{R} - \{2\}$ e. $[3, 5]$ f. $\mathbb{R} - [-2, 2]$ g. $[-\frac{5}{2}, -\frac{3}{4}]$ 11. – 6 12. 4 13. a. removable at $x = 3$, define $f(3) = 6$ b. removable at $x = 0$, define $f(0) = 0$ c. removable at $x = -1$, define $f(-1) = 3$ d. jump discontinuity at $x = 0$

14. a. jump discontinuity at $x = 0$ b. jump discontinuity at integer values c. removable at $x = 1$, define $f(1) = 3$

d. jump discontinuity at $x = 2$ e. jump discontinuity at $x = \frac{\pi}{2}$ f. jump discontinuity at $x = 0$ 15. max $= 11$, min $= -7$

16. a. $M = 2$, $m = 0$ b. $M = 3$, $m = 0$ c. $M = 1$, $m = \frac{1}{\sqrt{2}}$ d. $M = 5$, $m = 1$ e. $M = 28$, $m = 0$

17. a. $M = 14$, $m = -2$ b. $f(x)$ is continuous, so $f(0) = -2$, $f(4) = 14$ and $-2 < -1 < 0 < 11 < 14$

18. image $= [-\frac{29}{4}, 5]$, max $= 5$, min $= -\frac{29}{4}$ 19. a. $[-\frac{13}{4}, 27]$ b. $M = 27$, $m = -\frac{13}{4}$

20. Hint: use the Intermediate Value Theorem 21. Hint: use the Intermediate Value Theorem 22. a. $\frac{1}{3} < x < 3$

23. $m \in (-8, 8)$ 24. $x \in \mathbb{Z} \cup (-2, -1)$ 25. -5 26. at $x = 0$, the distance of the jump $= 9$

27. at $x = 1$, the distance of the jump $= 3$ 28. a. infinite discontinuity at $x = \frac{k\pi}{2}$

b. infinite discontinuity at $x = 1$ c. removable discontinuity at $x = 0$ d. infinite discontinuity at $x = \pm 1$

29. a. $a \in \{\frac{3}{2}, -1\}$ 30. $a = 2$, $b = -15$

ANSWERS TO TESTS

TEST 1A

1.	C	9.	B
2.	A	10.	D
3.	D	11.	A
4.	E	12.	C
5.	B	13.	B
6.	B	14.	D
7.	D	15.	E
8.	D	16.	B

TEST 1B

1.	D	9.	D
2.	E	10.	C
3.	C	11.	E
4.	C	12.	B
5.	D	13.	A
6.	B	14.	D
7.	A	15.	C
8.	C	16.	C

TEST 2A

1.	A	9.	C
2.	A	10.	E
3.	E	11.	E
4.	B	12.	B
5.	C	13.	A
6.	A	14.	C
7.	D	15.	E
8.	B	16.	A

TEST 2B

1.	E	9.	D
2.	E	10.	E
3.	C	11.	B
4.	B	12.	E
5.	B	13.	A
6.	C	14.	E
7.	D	15.	C
8.	C	16.	D

TEST 2C

1.	C	9.	B
2.	B	10.	D
3.	E	11.	A
4.	D	12.	B
5.	B	13.	D
6.	C	14.	E
7.	A	15.	B
8.	A	16.	D

TEST 3A

1.	C	9.	E
2.	B	10.	C
3.	E	11.	C
4.	A	12.	C
5.	C	13.	D
6.	A	14.	B
7.	B	15.	B
8.	E	16.	A

TEST 3B

1.	C	9.	B
2.	D	10.	C
3.	D	11.	D
4.	B	12.	A
5.	E	13.	C
6.	B	14.	E
7.	D	15.	C
8.	A	16.	D

GLOSSARY

A

absolute value: the non-negative difference of a number x and zero, written $|x|$. For example, $|8| = 8$ and $|-8| = 8$.

absolute value function: the function $|f(x)|$ defined as follows: if $f(x) \geq 0$ then $|f(x)| = f(x)$, and if $f(x) < 0$ then $|f(x)| = -f(x)$.

C

composite function: a function which is formed by composing two or more elementary functions, for example: $f(g(x))$.

constant function: a function of the form $f(x) = c$, where c is constant.

continuous function: a function whose graph contains no breaks or gaps.

continuous function at a point: a function which is defined at a point and whose right-hand and left-hand limits at the point are equal to the image of the point.

continuous function on an interval: a function which is continuous at every point on an interval.

crucial point: a point at which we need to check the right-hand limit and the left-hand limit.

D

decreasing function: A function $y = f(x)$ is a decreasing function on an interval I if y decreases as x increases on I. $f(x) = 6 - x$ is a decreasing function in $\mathbb{R}$.

discontinuous function: a function whose graph contains breaks or gaps.

discontinuous function at a point: a function which is not continuous at a given point.

domain of a function: the largest set of real x-values for which a function f is defined.

E

ε-neighborhood of a number: for $\varepsilon > 0$, the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is the ε-neigborhood of the number x_0.

even function: a function $f: D \to R$ for which $f(-x) = f(x)$ for all $x \in D$. The graph of an even function is symmetric with respect to the y-axis.

exponential function: For $a \in \mathbb{R}^+ - \{1\}$, the function $f: \mathbb{R} \to \mathbb{R}^+$, $f(x) = a^x$ is called the exponential function.

Extreme Value Theorem: Let f be a function defined from $[a, b]$ into $\mathbb{R}$. If f is continuous on $[a, b]$ then it has a maximum and a minimum on the interval $[a, b]$.

F

floor function: the function $[\![f(x)]\!]$ defined as follows: if $f(x) \in \mathbb{Z}$ then $[\![f(x)]\!] = f(x)$, and if $f(x) \notin \mathbb{Z}$ then $[\![f(x)]\!]$ is the greatest integer which is smaller than $f(x)$.

function: a rule that maps each element of a set D (called the domain) to a single element of a set R (called the range).

I

identity function: a function which is of the form $f(x) = x$.

image of a function: For a function $f: D \to R$, the image I (also called the image set) of f is the set of images of all the elements in D: $I \subseteq R$.

image of a point: For a function f and a point c, $f(c)$ is the image of c.

image set: see *image of a function*.

increasing function: A function $y = f(x)$ is an increasing function on an interval I if y increases as x increases on I. $f(x) = x + 6$ is an increasing function in $\mathbb{R}$.

indeterminate form: a limit of a function which is not defined or not bounded at a point.

infinite discontinuity: If at least one of the right-hand or left-hand limits of a function f at a point x_0 tends to infinity then f has infinite discontinuity at x_0.

Intermediate Value Theorem Let f be a function which is continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. Let k be any number between $f(a)$ and $f(b)$. Then there exists at least one real number $c \in (a, b)$ such that $f(c) = k$.

inverse function The inverse of a function $f: A \to B$ is the function $f^{-1}: B \to A$ such that $f^{-1}(f(x)) = x$ for every $f(x) \in B$. The inverse of f can only exist if f is a one-to-one and onto function.

J

jump discontinuity If the right-hand and left-hand limits of the function f are not equal at a point x_0 then f has jump discontinuity at x_0.

L

left-hand limit the limit of a function $f(x)$ at a point x_0 as x approaches x_0 from the left ($x < x_0$).

limit of a function Informally, a function f has a limit L at a point x_0 if the value of $f(x)$ gets closer and closer to L as x approaches x_0.

linear function a function which is of the form $f(x) = ax + b$ for $a, b \in \mathbb{R}$.

logarithmic function For $a \in \mathbb{R}^+ - \{1\}$, the function $f: \mathbb{R}^+ \to \mathbb{R}, f(x) = \log_a x$ is called the logarithmic function.

N

non-decreasing function A function $y = f(x)$ is a non-decreasing function on an interval I if y does not decrease as x increases on I.

non-increasing function A function $y = f(x)$ is a non-increasing function on an interval I if y does not increase as x increases on I.

O

odd function a function $f: D \to R$ for which $f(-x) = -f(x)$ for all $x \in D$. The graph of an odd function is symmetric with respect to the origin.

one-sided limit the right-hand limit or left-hand limit of a function at a point.

onto function A function $f: A \to B$ is an onto function if for any $y \in B$ there exists an $x \in A$ such that $f(x) = y$.

one-to-one function A function $f: A \to B$ is a one-to-one function if for each $x_1 \neq x_2$ in A, $f(x_1) \neq f(x_2)$.

P

periodic function a function f which satisfies $f(x + t) = f(x)$ for some $t \in \mathbb{R}$.

piecewise function a function which is defined by different formulas in different intervals of its domain.

polynomial function a function of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where n is a positive integer and $a_n \neq 0$.

R

radical function a function which contains one or more radical expressions such square roots, cube roots, etc.

range (of a function) the set which includes at least all the images of the elements in the domain of a function f.

rational function a function of the form $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

removable discontinuity A function f has removable discontinuity at a point x_0 if the limit of the function f at x_0 exists but is not equal to $f(x_0)$.

right-hand limit the limit of a function $f(x)$ at a point x_0 as x approaches x_0 from the right ($x > x_0$).

S

sign function the function $\mathrm{sgn}(f(x))$ defined as follows: if $f(x)$ is positive then $\mathrm{sgn}(f(x))$ has value 1, if $f(x)$ is 0 then $\mathrm{sgn}(f(x)) = 0$ and when $f(x)$ is negative, $\mathrm{sgn}(f(x)) = -1$.

square root function a function of the form $f(x) = \sqrt{g(x)}$.